MATHEMATICS PROBLEMS WITH SEPARATE PROGRESSIVE SOLUTIONS: HINTS, ALGORITHMS, PROOFS

VOLUME 1: INTERMEDIATE AND COLLEGE ALGEBRA

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INTRODUCTION

Problem-solving is a necessary activity not only for students within the didactic process and for acquiring mathematical proficiency, but also as a primary form of mathematical research and creation.

Proposing and solving problems represent continual scientific training, which the future mathematician begins in classes and continues through his or her individual preparation for school and contests, having as the main goal self-improvement in mathematics, perhaps even pursuing a career in this field.

For problem solvers, this ongoing scholastic training offers the principal way of developing the skills essential to mathematical thinking and technique, as well as achieving some successes in competition. These necessary skills include intuition, selective observation ability, analytical approach, theoretical framing, deduction and logical construction ability, and calculation speed.

However, the didactic material necessary for such training – namely problem books – while abundant in the publication market, for the most part do not reach a professional level which would allow the user to develop those previously enumerated skills.

In considering problem books with solutions, we observe that many of these books remain in the status of general collections of problems and solutions, and are presented as such. Not only do these books not provide solving methodologies, much less systematized methodologies, but also they lack methodic criteria or even affiliation with a certain domain or subdomain of mathematics. Moreover, those problem books do not provide a structure by which to stimulate and enrich individual study so necessary to the mathematical development of students.

In launching this publishing project – a series of problem books in a new structural format – we hope to intervene actively in the processes of exploring, attempting, and effectively solving problems. Our structure moves logically from the approach and theoretical framing of problems, through the steps to be executed, and finally to the generation of a complete solution.

The problem book is structured in four separate and independent sections, namely *Problems*, *Hints*, *Algorithms*, and *Proofs*, in this order.

The *Problems* section consists of 101 problems themselves, which are of medium to advanced difficulty level. These problems were selected from the category of those whose solution is based exclusively on the elementary theoretical results learned in classes.

The solution, however, does not fall into place as a direct application of the theory, but rather, they are the outcome of a non-trivial process of deduction, construction, and observation. These problems are specific to out-of-class mathematics workshops and preparation for mathematics contests.

Each problem has a corresponding position in each of the three sections coming next, which actually contain the progressive solutions to the problem.

The *Hints* offer groups of keywords (which can be words, groups of words, sentences, or short mathematical expressions) that suggest to the solver – intuitively, as well as analytically – an initial approach to the problem, important observations upon which the solution is based, categories of theoretical results applied when solving, and specific theoretical results.

The hints also suggest indirectly the solving algorithm (found in the next section), but without exposing or synthesizing it. All these suggestions, indications, and references are presented in an incomplete, short form, leaving to the solver the task of investigating the various possible approaches and choosing the one that leads to the correct solution.

In the *Algorithms* section, the solving algorithms provide chronological groups of steps necessary for generating the complete solution.

The algorithm is presented as a brief list of tasks; it does not reveal the complete solution to the problem, but only points out the partial tasks whose results will finally yield the logical construction of the solution. These tasks are shown in a short form, without presenting explicitly the mathematical objects upon which to operate, but rather, making precise references to the subjects of the previous steps.

The algorithm indicates directly the correct path for solving the problem, as well as the methodology associated with each step, but does not expose the concrete work, the way of combining the partial results for building the logical constructions, or the detailed development of the calculations.

The reason for this deliberate abbreviation is to allow the solver to discover the exact identification of the objects and working tasks, with the aim of integral reconstitution of the solution.

These features eliminate the risk of a wrong approach to the problem or of following an incorrect or nonproductive path toward a solution; at the same time, they leave enough room for the individual to work toward completing the integral solution.

The *Proofs* represent the complete integral solutions of the problems, unfolded according to the solving algorithm.

This comprehensive presentation includes the detailed steps to be executed, the observations that precede the deductions, and the entire logical motivation. No partial results are left unproved, neither as an exercise nor as being obvious or easily deduced.

Thus the material can be read, followed, and understood by as large a category of problem solvers as possible, not just those with an advanced level of mathematical training.

The sections described above are separated in this book so that the solver can explore the problem and search for solving paths independently, consulting the next section only when he or she has exhausted, with no success, his or her own approach and individual study methods.

As the solver moves progressively from a partial solution to a more complete one, this additional effort itself becomes a useful mathematical exercise. Moreover, the process of moving successively through the indications of the problems together with individual investigation and autocorrection of a wrong approach, stimulates and motivates the solver toward a solution.

All these elements give this type of problem book a truly interactive character.

The problems can be presented and discussed at mathematics workshops and sessions of preparation for contests and olympiads as well as in the classroom, using problems of different difficulty levels with separate groups of students; for advanced groups, an instructor can use this problem book in its entirety.

These problems offer a range of difficulty levels, from non-difficult problems, whose paths and methods of solving can be deduced directly from their wording, to quite difficult problems, and even to a difficulty level of international olympiad problems. Most of the solutions are not immediately obvious and do not represent direct applications of an isolated theoretical result, the theoretic ensemble being necessary for more complex solutions.

Solvers will also encounter problems of the “false difficult” type in which the statement of the problem creates the false impression of a long and laborious solution, while in fact the solution becomes visible as result of observing an important detail or making an ingenious construction or choice. Problems of this type should provide excellent counterexamples for those who tend to label a mathematics problem “impossible” when they fail to discover its solution immediately – and even to attach this label to mathematics itself.

The topics of the problems belong to algebra II and algebra of the first two years of college, passing through fields of integer and real numbers, equations, inequalities, powers, logarithms, divisibility, polynomials, and combinatorics. The problems were selected with care so that each problem meets all the intended methodological criteria, including the difficulty level.

The problem book actually contains three categories of problems, based on their difficulty level. These are arranged consecutively, passing gradually from one level to the next, so that the whole block of proposed problems is a homogeneous one.

The material, by its structure, is useful not only for problem solvers, but also for mathematics instructors, offering a didactic tool that can facilitate the development of students’ intuition in approaching medium and advanced level problems, as well as perfecting their algorithmic solving skills.

The problem-solving assumes theoretical and analytical skills, as well as algorithmic skills, coupled with a basic mathematical intuition. The concept of such a problem book successfully supports the development of these skills of the solver and meanwhile offers mathematics instructors models for teaching problem-solving as an integral part of the mathematics learning process.

The present work is the first of a series that will also operate in other domains and subdomains of mathematics. The problem books which follow will be presented in the same structural format, with separate progressive solutions.

This series is part of a far-reaching publishing project whose goal is the involvement of mathematics instructors and graduates in editing such interactive problem books, in similar or different formats, thereby enriching the supplementary didactic material so necessary to study and improvement in contest and scholarly mathematics.

PROBLEMS

AL1.1.1 Solve the system for x, y real:

$$(x-1)(y^2+6)=y(x^2+1)$$
$$(y-1)(x^2+6)=x(y^2+1)$$

AL1.1.2 Consider the sequence of positive integers that satisfies $a_n = a_{n-1}^2 + a_{n-2}^2 + a_{n-3}^2$ for all $n \geq 3$. Prove that if $a_k = 1997$, then $k \leq 3$.

……………………. Missing part ………………………

AL1.1.5 Prove that the equation $x^2 + y^2 + z^2 + 3(x+y+z) + 5 = 0$ has no solutions in rational numbers.

AL1.1.6 Given that $133^5 + 110^5 + 84^5 + 27^5 = k^5$, with k an integer, find k.

……………………. Missing part ………………………

AL1.1.77 Let $x_1, x_2, \ldots, x_n$ be n non-zero numbers. Prove that there exists an irrational number a such that numbers $ax_i, i = 1, \ldots, n$ are all irrational.

*N. Ceti, V. Marchida*n

AL1.1.78 Show that number $N = 1 + 2\cdot 3 + 4\cdot 5\cdot 6 + 7\cdot 8\cdot 9\cdot 10 + 11\cdot 12\cdot 13\cdot 14\cdot 15 + \cdots$ is not perfect square, if the number of its terms is larger than 1.

C. Rusu

AL1.1.79 Find the remainder of number 5^{7^n}, $n \in N$, upon division by 31.

D. Andrica

AL1.1.80 Prove that $\left|\sin 2x + 2\sin(x+y) - \sin 2y\right| \le 2\sqrt{2}$, for all reals x, y.

D. Bătinețu

AL1.1.81 Given 1000 non-zero and distinct natural numbers having the sum 1000998, prove that there are at least two odd numbers among them.

L. Niculescu

……………………. Missing part ………………………

AL1.1.95 Find all polynomials $P \in R[X]$ such that $P(x^2) = P^2(x), \forall x \in R$.

AL1.1.96 Show that whatever five integers we choose, there are two among them, whose sum or difference is divisible by 7.

I. Tomescu

……………………. Missing part ………………………

AL1.1.101 Let P be a polynomial with complex coefficients. Prove that its associated polynomial function is even if and only if there exists a polynomial Q with complex coefficients such that $P(x) = Q(x)Q(-x), \ \forall x \in C$

M. Țena

HINTS

AL1.1.1 Adding and subtracting the equations; completing squares; substitutions.

AL1.1.2 Reductio ad absurdum; the given relation holds for two different triples of four consecutive terms; inequalities between numbers that differ each other by a positive quantity.

…………………… Missing part ……………………

AL1.1.5 Sum of squares; substitutions; reductio ad absurdum; every perfect square is congruent to 0 or 1 modulo 4; theorem of division; divisibility; parity.

AL1.1.6 The last digits of n and n^5 are the same; the left-hand side is congruent to 0 modulo 3; inequalities.

…………………… Missing part ……………………

AL1.1.77 If m is prime, then $\sqrt{m}$ is irrational; table with n columns and an infinity of rows; reductio ad absurdum; counting principle; a, b rational implies a/b rational.

AL1.1.78 Terms multiples of 10; remainder upon division by 10; last digit.

AL1.1.79 5^{3k+1}; $125 = 124 + 1$.

AL1.1.80 Formula of (sin a – sin b); division by $\sqrt{1+\sin^2(x-y)}$; properties of functions sin and cos.

…………………… Missing part ……………………

AL1.1.95 $P = 0$, $P = 1$; writing P in its standard form; $P = a_n X^n$; reductio ad absurdum; if polynomials P and Q are equal and P contains a term in X^m , then Q also contains such a term.

AL1.1.96 Remainders of the perfect squares upon division by 7; theorem of division; counting principle; formula of difference between two squares; divisibility.

……………………. Missing part ……………………….

AL1.1.101 Standard form of P; identification of coefficients; $P(x) = P_1(x^2)$; roots of P_1; canonical form of P_1; formula of difference between two squares.

ALGORITHMS

AL1.1.1

Add the two given equations together.

Complete squares in the resulting equation.

Subtract the second given equation from the first.

Group the terms and factor out to get
$(x-y)(x+y-2xy+7)=0$.

Rearrange the terms and factor out in the second factor of the product.

Do the substitutions $a = x - 5/2$ and $b = y - 5/2$.

Solve the new system in a and b formed by the last equation and the one obtained after adding together the two given equations.

Arrange the terms such to put in evidence $a + b$.

Solve the quadratic equation in $a + b$.

Do a new subtraction and arrange the terms to put in evidence $a - b$.

Find $a - b$.

AL1.1.2

Reductio ad absurdum. Assume $k > 3$ for which the given relation holds.

Consider the four consecutive terms that must exist.

Apply the given relation for the fourth term $a_k = 1997$ and show that $a_{k-1} \le 44$.

Apply the given relation for the third term a_{k-1} and show that $a_{k-1} \ge 61$, contradiction.

……………………. Missing part ……………………….

AL1.1.5

Complete squares and write the left side of the equation as a sum of squares.

Show that the given equation is equivalent with
$a^2 + b^2 + c^2 = 7d^2$ in integers.

Reductio ad absurdum: assume a minimal solution (in sense of the sum of the absolute values).

Show that every perfect square is congruent to 0 or 1 modulo 4, using the theorem of division.

Show that a, b, c, d are even.

Show that (a/2, b/2, c/2, d/2) is also a solution, which contradicts the induction hypothesis of having a minimal solution.

AL1.1.6

Show that the last digit of the left-hand side is 4.

Show that $3 \mid k$ and the smallest possibility for k is 144, the next is 174.

Find that each power from the left-hand side is smaller than a multiple of 10^{10} and add them together.

Show that the left-hand side is smaller than 10^{11}.

Show that k cannot be 170, hence it cannot be 174.

…………………… Missing part ………………………

AL1.1.77

Show that all numbers $\sqrt{m}$, with m prime, are irrational.

Consider all numbers $\sqrt{m}$, with m prime, and denote them by $y_1, y_2, \ldots, y_m, \ldots$. Build the infinite table $\left(y_i x_j\right)_{i=1,\infty}^{j=1,n}$ with n columns and an infinity of rows. Show that there exists a row containing only irrational numbers:

Prove by reductio ad absurdum: Assume that each row holds at least one rational number. Show that there exists at least one column holding two rational numbers.

Show that the ratio of these two numbers is irrational, contradiction.

AL1.1.78

Show that all terms from the fourth upward are multiples of 10.

Show that the last digit of N is 7.

Reductio ad absurdum: Assume N is a perfect square. Show that the last digit of N cannot be 7, contradiction.

AL1.1.79

Show that $5^{7^n} = 5^{3k+1}$, writing 7 = 6 + 1.

Show that $5^{3k+1} = (31m+1)^k \cdot 5$, writing 125 = 124 + 1.

Show that $5^{7^n} = 31q + 5$.

AL1.1.80

Denote by $f(x, y)$ the expression within the modulus. Express $f(x, y)$, writing $\sin 2x - \sin 2y$ as a product and factoring out $t = \sqrt{1+\sin^2(x-y)}$.

Show that there exists $g(x, y)$ real such that $\sin g(x,y) = \dfrac{\sin(x-y)}{t}$ and $\cos g(x,y) = \dfrac{1}{t}$.

Replace these expressions in the expression of $f(x, y)$ and show that $|f(x, y)| \le 2\sqrt{2}$, using $|\sin a| \le 1, \forall a \in R$.

…………………….. Missing part ………………………

AL1.1.95

Check that polynomials $P = 0$ and $P = 1$ satisfy the given condition.

Assume $P \ne 0, P \ne 1$. Show that P is of the form $a_n X^n$ by reduction ad absurdum:

Write P in its standard form, with coefficients a_i and degree n.

Assume that there exists $k \in \{0, 1, \ldots, n-1\}$ such that $a_k \ne 0$ and $a_s = 0$ for any $s \in \{k+1, k+2, \ldots, n-1\}$. Write the given relation by replacing the coefficients according to this assumption.

Study the terms in x^{n+k} in both sides and show that such a term does not exist in the left-hand member, contradiction.

Write the given equality for $P = a_n X^n$ and show that $a_n = 1$.

AL1.1.96

Show that the possible remainders of a perfect square upon division by 7 are 0, 1, 2, 4, using the theorem of division.

Consider five integers and their squares.

Apply the previously proven property to these squares.

Show that at least two of these squares have the same remainder upon division by 7, using the counting principle.

Express the difference of these two squares and show that the sum or the difference of their square roots is divisible by 7.

…………………… Missing part ………………………

AL1.1.101

Check that the implication "$\Leftarrow$" is obvious ($P(x) = P(-x)$).

For the implication "$\Rightarrow$" assume function P is even and write the relation $P(x) = P(-x)$ using the standard form of polynomial P, with coefficients a_i and degree n.

Show that all coefficients of the terms with odd powers of x are null.

Show that P can be written as $P(x) = P_1(x^2)$, for any real x.

Consider numbers $y_i \in C$ such that $y_i^2 = x_i$ $(i = 1, \ldots, n)$ and $b \in C$ such that $b^2 = (-1)^n a$ and rewrite the relation obtained previously.

Group the factors conveniently, use the formula of difference between two squares and express $P(x)$ in the form $Q(x)Q(-x)$.

PROOFS

AL1.1.1

We add together the two given equations. After simplifying the resulting equation and completing the squares, we arrive at the following equation: $(x-5/2)^2+(y-5/2)^2=1/2$. (1)

We subtract the second given equation from the first and group the terms:

$$xy(y-x)+6(x-y)+(x+y)(x-y)=xy(x-y)+(y-x)$$
$$(x-y)[-xy+6+(x+y)-xy+1]=0$$
$$(x-y)(x+y-2xy+7)=0$$

Thus, either $x-y=0$ or $x+y-2xy+7=0$. The only ways to have $x-y=0$ are with $x=y=2$ or $x=y=3$ (found by solving equation (1) with the substitution $x=y$). Now, all solutions to the original system where $x \neq y$ will be solutions to $x+y-2xy+7=0$. This equation is equivalent to the following equation (derived by rearranging terms and factoring): $(x-1/2)(y-1/2)=15/4$. (2)

Now we solve equations (1) and (2) simultaneously.

Let $a=x-5/2$ and $b=y-5/2$. Then, equation (1) is equivalent to $a^2+b^2=1/2$. (3)
and equation (2) is equivalent to:

$$(a+2)(b+2)=15/4 \Rightarrow ab+2(a+b)=-1/4$$
$$\Rightarrow 2ab+4(a+b)=-1/2. \quad (4)$$

Adding equation (4) to equation (3), we find:

$$(a+b)^2+4(a+b)=0 \Rightarrow a+b=0,-4. \quad (5)$$

Subtracting equation (4) from equation (3), we find:

$$(a-b)^2-4(a+b)=1. \quad (6)$$

Observe that if $a+b=-4$, then equation (6) will be false. Thus, $a+b=0$. Substituting this into equation (6), we obtain:

$$(a-b)^2=1 \Rightarrow a-b=\pm 1 \quad (7)$$

Since $a+b=0$, we now can find all ordered pairs (a, b) with the help of equation (7). They are $(-1/2, 1/2)$ and $(1/2, -1/2)$.

Therefore, the only solutions (x, y) are $(2, 2)$, $(3, 3)$, $(2, 3)$, and $(3, 2)$.

AL1.1.2

Assume that for some $k > 3$, $a_k = 1997$. Then, each of the four numbers a_{k-1}, a_{k-2}, a_{k-3}, and a_{k-4} must exist.

Let $w = a_{k-1}$, $x = a_{k-2}$, $y = a_{k-3}$, and $z = a_{k-4}$.

By the given condition, $1997 = w^2 + x^2 + y^2$. Thus, $w \le \sqrt{1997} < 45$ and since w is a positive integer, $w \le 44$.

But then $x^2 + y^2 \ge 1997 - 44^2 = 61$.

Now, $w = x^2 + y^2 + z^2$. Since $x^2 + y^2 \ge 61$ and $z^2 \ge 0$, $w = x^2 + y^2 + z^2 \ge 61$. But $w \le 44$, contradiction.

…………………… Missing part ………………………

AL1.1.5

Let $u = 2x + 3$, $v = 2y + 3$, $w = 2z + 3$. Then the given equation is equivalent to $u^2 + v^2 + w^2 = 7$.

Asking that the above equation has solutions in rational numbers is equivalent to ask that the equation $a^2 + b^2 + c^2 = 7d^2$ has non-zero solutions in integers. Assume on the contrary that (a, b, c, d) is a non-zero solution with $|a| + |b| + |c| + |d|$ minimal.

We show first that every perfect square is congruent to 0 or 1 modulo 4. Indeed, if $n = 4m + k$ with $k \in \{0, 1, 2, 3\}$, then $n^2 = 16m^2 + 8mk + k^2$ and one can easily check that for $k = 0, 1, 2, 3$, the remainders of k^2 upon division by 4 can be only 0 or 1. Hence the possible remainders of n^2 upon division by 4 are still 0, 1. Thus we proved that every perfect square has this property.

Under the condition of the solution being minimal, we have that each of a, b, c, d is congruent to 0 modulo 4.

Thus, we must have a, b, c, d even, therefore d is also even.

But then (a/2, b/2, c/2, d/2) is also a solution of our equation and is a smaller solution, which is a contradiction.

AL1.1.6

The last digits of n and n^5 are the same. Hence, the last digit of the left-hand side is the same as that of $3 + 0 + 4 + 7$, which is 4.

Hence, the last digit of k is *4*. Also $133 \equiv 1 \pmod 3$, $110 \equiv -1 \pmod 3$, $84 \equiv 0 \pmod 3$, $27 \equiv 0 \pmod 3$, so the left-hand side is congruent to 0 modulo 3. Obviously, $k > 133$.

Therefore, the smallest possibility is 144, the next is 174.

Now

$11^5 = (10+1)^5 = 10^5 + 5 \cdot 10^4 + 10 \cdot 10^3 + 10 \cdot 10^2 + 5 \cdot 10 + 1 = 161051$,

so $110^5 = 11^5 \cdot 10^5 = 1.61051 \cdot 10^{10} < 2 \cdot 10^{10}$.

Obviously, 27 and 84 are smaller than 100, so 27^5 and 84^5 are smaller than 10^{10}.

Similarly, $133^5 < (1331/10)^5 = 11^{15}/10^5 < 5 \cdot 10^{10}$.

Hence, the left-hand side is smaller than 10^{11}. But $170^2 = 28900 > 28000$, $170^4 = 780000000 > 7 \cdot 10^8$ and $170^5 > 10^{11}$.

Hence the only possibility for k is 144.

……………………. Missing part ……………………….

AL1.1.77

The numbers of form $\sqrt{m}$, with m prime, are irrational.

Indeed, for $m = 1$ the affirmation is true. For $m > 1$, if we assume by absurdum that $\sqrt{m} = \frac{a}{b}$, with a, b integers, and we take fraction a/b in lowest terms, it follows that $mb^2 = a^2$ and hence $m \mid a^2$ and, since m is prime, we deduce that $m \mid a$.

Therefore $a = mk$, with k natural. Replacing back, we obtain $mb^2 = m^2k^2$, that is $b^2 = mk^2$.

By a similar argument, we obtain $m \mid b$. Hence $m > 1$ is a common divisor of numbers a and b, contradiction.

Denote these numbers $\sqrt{m}$ by $y_1, y_2, \ldots, y_m, \ldots$. We build the following infinite table:

$$\begin{array}{ccccc} a_1x_1 & a_1x_2 & \dots & a_1x_{n-1} & a_1x_n \\ a_2x_1 & a_2x_2 & \dots & a_2x_{n-1} & a_2x_n \\ \dots & \dots & \dots & \dots & \dots \\ a_mx_1 & a_mx_2 & \dots & a_mx_{n-1} & a_mx_n \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

We prove the required property by reductio ad absurdum. Assume that each row holds at least one rational number.

Since we have an infinity of rows and only *n* columns, according to counting principle it follows that there exists at least one column holding two rational numbers. Let this be the column with elements x_q. The two rational numbers would be of form $a_i x_q$ and $a_j x_q$. Making their ratio, we get $\frac{a_i x_q}{a_j x_q} = \frac{a_i}{a_j}$.

But since a_i and a_j are irrational and the numbers under the radicals are prime, it follows that their ratio is irrational. We arrived at a contradiction, namely that a rational number is equal to an irrational one.

AL1.1.78

All terms with the rank higher than 3 are multiples of 10, because the product contains at least one even number and one number divisible by 5.

It follows that the remainder of *N* upon division by 10 is $1+2\cdot 3=7$ and hence 7 is the last digit of *N*.

But there is no perfect square whose last digit is 7, because the perfect squares can only have 0, 1, 4, 5, 6 or 9 as their last digit.

AL1.1.79

We can write number 5^{7^n} in the following forms:

$5^{7^n} = 5^{(6+1)^n} = 5^{3k+1} = 125^k \cdot 5 = (31m+1)^k \cdot 5 = (31p+1)\cdot 5 = 31q+5$.

Hence the remainder of the given number upon division by 31 is 5.

AL1.1.80

Let $f(x, y) = \sin 2x + 2\sin(x+y) - \sin 2y =$

$= 2\sin(x-y)\cos(x+y) + 2\sin(x+y) =$

$$= 2\sqrt{1+\sin^2(x+y)}\left(\frac{\sin(x-y)}{\sqrt{1+\sin^2(x+y)}}\cos(x+y) + \right.$$

$$\left. + \frac{1}{\sqrt{1+\sin^2(x+y)}}\sin(x+y)\right).$$

But $\left(\frac{\sin(x-y)}{\sqrt{1+\sin^2(x+y)}}\right)^2 + \left(\frac{1}{\sqrt{1+\sin^2(x+y)}}\right)^2 = 1$ and hence there exists $g(x, y) \in R$ such that:

$$\frac{\sin(x-y)}{\sqrt{1+\sin^2(x+y)}} = \sin g(x, y) \quad \text{and} \quad \frac{1}{\sqrt{1+\sin^2(x+y)}} = \cos g(x, y)$$

Replacing them in the found expression of $f(x, y)$ yields:

$$|f(x, y)| = 2\sqrt{1+\sin^2(x-y)}\,|\sin(x+y+g(x, y))| \le$$

$$\le 2\sqrt{1+\sin^2(x-y)} \le 2\sqrt{2}.$$

.... Missing part ....

AL1.1.95

One can easily check that polynomials $P = 0$ and $P = 1$ satisfy the given condition. Assume now $P \ne 0, P \ne 1$. Let $P \in R[X]$, $P = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$, $a_n \ne 0$.

$$P(x_2) = P^2(x), \forall x \in R \Leftrightarrow a_n x^{2n} + a_{n-1}x^{2(n-1)} + \cdots + a_1 x^2 + a_0 =$$

$$= \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)^2, \forall x \in R$$

Assume that there exists $k \in \{0, 1, \ldots, n-1\}$ such that $a_k \ne 0$ and $a_s = 0$ for any $s \in \{k+1, k+2, \ldots, n-1\}$, which is equivalent to the fact that P is not of the form $a_n X^n$. Then we have:

$$P(x) = a_n x^n + a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$$

$$P(x^2) = a_n x^{2n} + a_k x^{2k} + a_{k-1}x^{2(k-1)} + \cdots + a_1 x^2 + a_0$$

The given relation becomes:

$$a_n x^{2n} + a_k x^{2k} + \cdots + a_1 x^2 + a_0 = \left(a_n x^n + a_k x^k + \cdots + a_1 x + a_0\right)^2, \forall x \in R$$

In the right-hand member of the above relation, the coefficient of x^{n+k} is $2a_n a_k \neq 0$, while in the left-hand member a term in x^{n+k} does not exist, since $2n > n + k > 2k$. Thus we arrived at a contradiction, so the assumption we made is absurd. Hence $P = a_n X^n$. Replacing back in the given relation yields:

$$P(x^2) = P^2(x), \forall x \in R \Leftrightarrow a_n x^{2n} = a_n^2 x^{2n}, \forall x \in R \Leftrightarrow a_n^2 = a_n \Rightarrow a_n = 1$$

Hence $P = X^n$. Therefore, the polynomials satisfying the given condition are $P \in \left\{0, 1, X^n\right\}$.

AL1.1.96

We show first that the possible remainders of a perfect square upon division by 7 are 0, 1, 2, or 4. Indeed, if $n = 7m + k$ with $k \in \left\{0, 1, 2, 3, 4, 5, 6\right\}$, then $n^2 = 49m^2 + 14mk + k^2$ and one can easily check that for $k = 0, 1, 2, 3, 4, 5, 6$ the remainders of k^2 upon division by 7 can be only 0, 1, 2, or 4. Hence the possible remainders of n^2 upon division by 7 are still 0, 1, 2, or 4. Thus we proved that every perfect square has this property.

Now let a, b, c, d, e be five integers. According to the above property, their squares can only have one of the remainders 0, 1, 2, or 4 upon division by 7. Since we have five numbers and only four remainders, according to the counting principle, there exist at least two squares that have the same remainder upon division by 7. Thus there exist $x, y \in \left\{a, b, c, d, e\right\}$ such that $x^2 - y^2 \vdots 7$. However $x^2 - y^2 = (x - y)(x + y)$ and since 7 is prime number, it follows that 7 divides $x + y$ or $x - y$.

……………………. Missing part ………………………

AL1.1.101

If there exists a polynomial Q with the stated property, then $P(-x) = Q(-x)\, Q(x) = P(x)$, so polynomial function P is even.

Conversely, assume that polynomial function P is even. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, with $a_i \in C, \forall i = 1, \ldots, n$.

Writing $P(x) = P(-x)$, $\forall x \in C$ and identifying the coefficients, it follows that all coefficients of the terms with odd powers of x are null. Hence we can write:

$$P(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n} = P_1(x^2),\ \forall x \in C, \quad (1)$$

where P_1 is a polynomial of degree n.

If $x_1, x_2, \ldots, x_n \in C$ are the roots of polynomial P_1, we can write $P_1(x) = a(x - x_1)(x - x_2)\cdots(x - x_n)$ with $a \in C^*$ and then (1) becomes $P(x) = a(x^2 - x_1)(x^2 - x_2)\cdots(x^2 - x_n) =$

$= (-1)^n a(x_1 - x^2)(x_2 - x^2)\cdots(x_n - x^2)$. (2)

Let $y_1, y_2, \ldots, y_n \in C$ be complex numbers such that $y_i^2 = x_i$ $(i = 1, \ldots, n)$ and let $b \in C$ be a complex number such that $b^2 = (-1)^n a$. Relation (2) becomes:

$$P(x) = b^2(y_1^2 - x^2)(y_2^2 - x^2)\cdots(y_n^2 - x^2) =$$

$$= [b(y_1 + x)(y_2 + x)\cdots(y_n + x)]\cdot[b(y_1 - x)(y_2 - x)\cdots(y_n - x)].$$

Taking $Q(x) = b(y_1 + x)(y_2 + x)\cdots(y_n + x)$, we have $P(x) = Q(x)Q(-x),\ \forall x \in C$.

References

American Mathematical Monthly, Vol. 116, Number 10, December 2007, USA

American Mathematical Monthly, Vol. 115, Number 6, June-July 2008, USA

Kvant, Number 3, 1970, URSS

Kvant, Number 11, 1971, URSS

Kvant, Number 2, 1978, URSS

Gazeta Matematică, Number 4, 1977, Romania

Gazeta Matematică, Number 5, 1977, Romania

Gazeta Matematică, Number 12, 1977, Romania

Gazeta Matematică, Number 3, 1979, Romania

Gazeta Matematică, Number 8, 1979, Romania

Gazeta Matematică, Number 1, 1980, Romania

Gazeta Matematică, Number 8, 1980, Romania

Gazeta Matematică, Number 11, 1980, Romania

Gazeta Matematică, Number 4, 1981, Romania

Gazeta Matematică, Number 5, 1981, Romania

Gazeta Matematică, Number 9, 1981, Romania

Gazeta Matematică, Number 10, 1981, Romania

N. Teodorescu, A. Constantinescu, M. Mihai, L. Pârşan, E. Perjariu, A. Popescu-Zorica, P. Radovici-Mărculescu, M. Țena, *Problems from Gazeta Matematică – Selective and Methodological Edition (Probleme din Gazeta Matematică – Ediție selectivă şi metodologică)*. Editura tehnică, Bucharest, 1984.

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