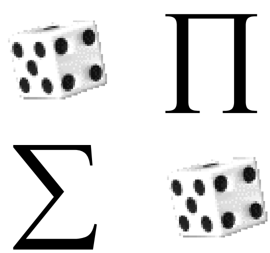
UNDERSTANDING AND CALCULATING THE ODDS

Probability Theory Basics and Calculus Guide for Beginners, with Applications in Games of Chance and Everyday Life



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Introduction

Every one of us uses the word *probable* few times a day in common speech when referring to the possibility of a certain event happening.

We usually say an event is *very probable* or *probable* if there are good chances for that event to occur.

This simplistic dictionary definition of the *probable* quality attached to an event is unanimously accepted in current speech.

Far from standing for a rigorous definition, this enunciation still gives evidence for the quantitative and measurement aspect of the probability concept because the *chance* of an event occurring is represented by figures (percentages).

Meanwhile, these chances are the numerical result of an estimation or calculation process that may start from various hypotheses.

You will see in the following chapters how the proper probability calculus can lead to a variety of numerical results for the same event. These results are a function of the initial information that is taken into account.

In addition, establishing a certain threshold from which the chance of an event occurring attribute to it the quality of being *probable* or *very probable* is a subjective choice.

All these elements create a first view of the relativity of the term *probable* and of the possible errors that can be introduced into the qualitative and quantitative interpretation of probability.

These interpretation errors, as well as that false *certainty* psychologically introduced by the numerical result of measuring an event, turn probability calculus into a somewhat dangerous tool in the hands of persons having little or no elementary mathematical background.

This affirmation is not at all hazardous, because probabilities are frequently the basis of decisions in everyday life.

We estimate, approximate, communicate and compare probabilities daily, sometimes without realizing it, especially to make favorable decisions.

The methods through which we perform these operations could not be rigorous or could even be incorrect, but the need to use probability as criterion in making decisions generally has a precedence.

This could be explained by the fact that human beings automatically refer to statistics in any specific situation, and statistics and probability theory are related.

We usually take a certain action as result of a decision because statistically, that action led to a favorable result in a number of previous cases. In other words, the probability of getting a favorable result after that action is acceptable.

This decisional behavior belongs to a certain human psychology and the human action is generally not conditioned by additional knowledge. Although statistics and even probability do not provide any precise information on the result of a respective action, the decision is made intuitively, without reliance on scientific proof that the decision is optimum.

For example, when we say, *"It will very probably rain today,"* the estimation of a big chance of rain results from observing the clouds or the weather forecast. Most of time it rains when such clouds are in the sky—according to statistics—therefore, it will very probably rain today, too. In fact, this is an estimation of the probability of rain, even though it contains no figures. If we take an umbrella, this action is the result of a decision made on the basis of a previous estimation. We choose to take an umbrella not because we see clouds, but because most of time it rains when the sky is cloudy.

When we say, *"It is equally probable to obtain 1 or 3 after rolling a die,"* we have observed that the die has six sides, and so the number of possible outcomes is six. Among those possible outcomes, one corresponds to occurrence of 1 and one to occurrence of 3. So, the chances are equal (1/6). Unlike the first example, the previous estimation of chances resulted from a much more rigorous observation, which led to a numerical result.

Another example—otherwise unwanted—of making a decision based on probability is the following:

Your doctor communicates the stages of evolution of your disease: if you won't have an operation, you have a 70 percent chance of living, and if you'll have the operation, you have a 90 *percent chance of a cure, but there is a 20 percent chance that you will die during the operation.*

Thus, you are in a moment when you have to make a decision, based on personal criteria and also on communicated figures (their estimation was performed by the doctor according to statistics).

In most cases of probability-based decisions, the person involved performs the estimation or calculus.

Here is a simple example:

You are in a phone booth and you must urgently communicate important information to one of your neighbors (let us say you left your front door open).

You have only one coin, so you can make only one call.

You have two neighboring houses.

Two persons live in one of them and three persons live in the other.

Both their telephones have answering machines.

Which one of the two numbers will you call?

The risk is that nobody will be at the home you call and the coin will be lost when the answering machine starts.

You could make an aleatory choice, but you could also make the following decision: "*Because the chances for somebody to be home are bigger in the case of house with three persons, I will call there*."

Thus, you have made a decision based on your own comparison of probabilities.

Of course, the only information taken into account was the number of persons living in each house.

If other additional information—such the daily schedules of your neighbors—is factored in, the probability result might be different and, implicitly, you might make a different decision.

In the previous example, the estimation can be made by anyone because it is a matter of simple counting and comparison.

But in most situations, a minimum knowledge of combinatorics and the calculus of probabilities are required for a correct estimation or comparison.

Millions of people take a chance in the lottery, but probably about 10 percent of them know what the winning probabilities really are.

Let us take, for example, the *6 from 49* system (six numbers are drawn from 49 and one simple variant to play has six numbers; you win with a variant having a minimum of four winning numbers).

The probability of five numbers from your variant being drawn is about 1/53992, and the probability of all six numbers being drawn (the big hit!) is 1/13983816.

For someone having no idea of combinations, these figures are quite unbelievable because that person initially faces the small numbers 5, 6 and 49, and does not see how those huge numbers are obtained.

In fact, this is the psychological element that a lottery company depends on for its system to work.

If a player knew those figures in advance, would he or she still play? Would he or she play less often or with fewer variants? Or would he or she play more variants in order to better the chances?

Whatever the answers to these questions may be, those probabilities will influence the player's decision.

There are situations where a probability-based decision must be made—if wanted—in a relatively short time; these situations do not allow for thorough calculus even for a person with a mathematical background.

Assume you are playing a classical poker game with a 52-card deck. The cards have been dealt and you hold four suited cards (four cards with same symbol), but also a pair (two cards with same value). For example, you hold 3♣ 5♣ 8♣ Q♣ Q♦. You must now discard and you ask yourself which combination of cards it is better to keep and which to replace.

To achieve a valuable formation, you will probably choose from the following two variants:

– Keep the four suited cards and replace one card (so that you have a flush); or

– Keep the pair and replace three cards (so that you have "three of a kind or better").

In this gaming situation, many players intuit that, by keeping the pair (which is a high pair in the current example), the chances for a *Q* (queen) to be drawn or even for all three replaced cards to have same value, are bigger than the chance for one single drawn card to be ♣ (clubs).

And so, they choose the “safety” and play for “three of a kind or better”. Other players may choose to play the flush, owing to the psychological impact of those four suited cards they hold.

In fact, the probability of getting a flush is about 19 percent and the probability of getting three of a kind or better is about 11 percent, which is about two times lower. In case you are aware of these figures beforehand, they may influence your decision and you may choose a specific gaming variant which you consider to have a better chance of winning.

This is a typical example of a decision based on probabilities in a relatively short time.

It is obvious that, even assuming you have probability calculus skills, it is impossible to calculate all those figures in the middle of the game. But you may use results memorized in anticipation obtained through your own calculations or picked from tables of guides containing collections of applied probabilities. In games of chance, most players make probability-based decisions as part of their strategy, especially regular players.

The examples shown thus far were not chosen randomly. They demonstrate the huge psychological component of our interaction with probabilities, especially in making decisions.

In addition to this practical aspect of interaction with chance and percentages, both laymen and scientists are fascinated by probability theory because it has multiple models in nature. It is a calculus tool for other sciences and the probability concept has major philosophical implications as well.

Returning to the immediate practical aspect, whether we have mathematical background or not, whether we know the precise definition of the concept or not, or whether we have calculus skills or not, many times we make decisions based on probabilities as a criterion, sometimes without realizing it.

But this criterion is not obligatory. We may use it as a function of intuition, as a personal principle of life, or among other subjective elements.

Those making no decisions based on figures are those who run the risk unconditioned by a certain threshold and, several times, the result is favorable for them.

But statistics show that probabilities are most often taken into account, whether in a simplistic or professional manner.

All these are well-founded reasons for creating a probability guide addressed to average people who have no solid mathematical background.

Although the main goal of this guide is practical, we also insisted on providing a rigorous review of probability concepts, whose lack makes the beginner subject to false interpretations and application errors.

The teaching material has been thus structured for people without a mathematical background to be able to picture a sufficiently clear view on the notions used and to solve the applications through an accurate framing of the problem and a correct use of calculus algorithms, even without reading the pure mathematics chapter titled *Probability Theory Basics.*

Those who feel that the high level mathematics is above their comprehension may skip that chapter without major repercussions on the practical goal, namely the correct application of calculus methods and algorithms.

This is possible because the presentation uses consistent references to mathematical notions through examples and natural models.

Moreover, the applications exclusively belong to finite probability fields, where the calculus algorithms and the reduced number of formulas used can be retained and applied without a complete and profound study of probability theory notions.

As a matter of fact, the reader needs only an elementary knowledge of mathematics from primary school to calculate odds and probabilities—operations with integer numbers and fractions (addition, subtraction, multiplication and division), the order of operations and elementary algebraic calculus.

Comfort with set theory and operations with sets is helpful, but all these notions can be found at the beginning of the chapter titled *Probability Theory Basics.* Also, some knowledge of combinatorics notions and formulas is a great advantage.

Most of gambling applications use combinations and many times the probability calculus reverts to counting them.

But the lack of such knowledge is successfully compensated for because the guide contains a solid chapter dedicated to this mathematical domain and the information provided there can be understood by anyone.

The structure and content of the next chapters follows:

What Is Probability?

Reading this chapter is essential in accommodating with the probability concept.

Starting from varying the mathematical definition, the probability is structurally shown as a measure and also as a limit.

All mathematical notions referred to, as well as the main probability theory theorems, are exposed through examples and natural models.

We also talk about the philosophical interpretations and implications of the concept and about the psychological impact of interaction of humans with probabilities.

Probability Theory Basics

This is the strictly mathematical chapter that contains all the rigorous definitions that establish the basis for the probability concept, starting from set operations, sequences of real numbers, convergence, Boole algebras, and measures, to field of events, probability, conditional probability and discrete random variables.

The chapter contains only the main theoretical results, which are presented as enunciations without demonstrations, but also contains many examples.

As we said before, reading this chapter is not obligatory to understanding the practical goal of calculus, but following it in parallels when running through the first chapter is useful for those having a minimal mathematical background.

Combinatorics

Combinatorial analysis is an important calculus tool in probability applications and this is the reason why it has a dedicated chapter.

This chapter contains the definitions of permutations, combinations and arrangements, along with their calculus formulas and main properties.

A lot of examples and solved and unsolved applications complete the theoretical part.

Beginner's Calculus Guide

This part of the work is in fact the main teaching material required by the beginner who wants to correctly apply probability calculus in practical situations.

The exposure of methods and applications is mostly algorithmic to enable the reader to easily follow the steps to be executed and to avoid the application errors.

Framing the problem, establishing the information to be taken into account, the correct enunciation of the events to measure, the adequate calculus method, the formulas to use, the calculus, all are explained and exemplified in depth and in an accessible manner.

Solved applications and suggestive examples are shown throughout this chapter.

You will also find here the most frequent errors in application or calculus pointed out to help beginners avoid them.

As in other chapters, the examples and solved applications mostly belong to games of chance, where the knowledge actually acquired has immediate application.

The probability calculus is explained and applied for the finite case (finite field of events), where the practical situations suitable for application are more numerous (in gambling and everyday life) and probability has a higher accuracy as a decisional criterion.

Probability Calculus Applications

This chapter verifies the theoretical and application knowledge acquired in the previous chapters.

It is in fact a collection of solved and proposed elementary problems that usefully and even necessarily complete the previous theoretical chapters. The difficulty level of the proposed problems grows gradually. The applications come from games of chance and also from everyday life.

WHAT IS PROBABILITY?

The goal of this chapter is to provide the reader who has no mathematical background with a sufficiently clear image of the probability concept, and how, in its absence, the approach of proper calculus becomes predisposed to errors, especially as they relate to the correct initial framing of problems.

This image should catch the basic structural aspects as well as properties that become manifest when we ascribe events to statistics and probability in daily life.

We will also talk about the psychological and philosophical implications of this concept, trying to delimit the mathematical term from the content of the common word *probability*.

Obviously, any explanation and presentation of the probability concept, even for people without a mathematical background, cannot overlook the mathematical definition. Thus, we will start with this definition and try to rebuild it step by step from its constituent parts. This rebuilding will be on the basis of particular cases and suggestive examples.

Such an attempt is a must because of the huge psychological impact of statistics and probabilities in people's daily lives.

Due to a natural need that is more or less rigorously justified, humans consistently refer to statistics; therefore, probability has become a real decision-making tool.

In this chapter we discuss in detail this major psychological component of the probability concept.

The fact that people estimate, calculate and compare probabilities with the purpose of making decisions, without knowing exactly the concept's definition or without mastering the proper calculus, automatically generates the risk of qualitative interpretation errors and the errors that come of using the figures as decision-making criterion as well.

Psychologically speaking, the tendency of novices to grant the word *probability* a certain importance is generally excessive in two ways: the word is granted too much importance—the figures come to represent the subjective absolute degree of trust in an event occurring—or too little importance—so many times, an equal sign is put between *probable* and *possible* and the information provided by numbers is not taken into account.

We use here the term *word* instead of *concept* deliberately when talking about probability.

To have a good understanding of what *probability* means and implies, we do not refer to the mathematical definition exclusively, but also to how this notion is perceived in a nonacademic environment for people having low or average knowledge level.

This approach is necessary because probability, as both a notion and a tool, has huge psychological implications when interacting with human concerns in daily life.

Words and concepts

For a correct interpretation of all information we collect to study the various objects of knowledge, is absolutely necessary to be able to make the distinction, so many times hardly discernible, between word and concept.

A *word* is a graphic and phonetic representation of a category of objects that are subjects of judgments or of human communication.

Thus, a word identifies a group of objects from the surrounding reality, whether physical, perceived or abstract.

For example, we can assign to the word *apparatus* the set of objects (television, radio, microphone, computer, the group of internal human organs that work the digestion, and so on). (This is in fact a set of words, which, by an abuse of denotation, has been presented as set of physical objects, namely, the set of all such kinds of objects existing). The denotation *apparatus* represents that entire set.

As another example, we can assign to the word *ball* the set of objects {the round object used in soccer, the round mobile object in pool (snooker) game, a celestial body, and so on}, whose representative is the word itself.

The word is practically a symbol, a denotation that is assigned to the set of objects it represents.

Language arose and evolved during history along with the need for human communication, but the status of a word always remains the same: a symbol indispensable to communication but deprived of conceptual content.

The word is not an object itself, but a symbolic representation of other objects from the surrounding reality, having the exclusive purpose of communication.

Words are used in common nonacademic language as communication symbols. The transmission of a certain word automatically refers to the entire set of objects that word represents.

These sets of objects are generally accepted by the community using a specific language, in the sense of majority.

This agreement is neither written nor officially stated by a certain organization invested in this matter, nor is it suggested by any academic society. While most people refer to dictionaries to gain the definition of a word to avoid argument about its usage, in this case, it is the practical result of free human communication throughout history.

A *definition* is a grammatical sentence delimiting or extending a set of objects that can be attached to a word by enunciating the properties specific to the objects it describes.

A definition can be attached to a word (or group of words) when the set of objects it represents is not unanimously accepted or there are doubts about this matter within a community or even between two interlocutors.

On the other hand, a definition can be attached to a group of objects in order to simplify the ulterior communications that refer to those objects.

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The concept of probability

Initially, probability theory was inspired by games of chance, especially in 17th century France, and was inaugurated by the Fermat–Pascal correspondence.

However, its complete axiomatization had to wait until Kolmogorov’s *Foundations of the Theory of Probability* in 1933.

Over time, probability theory found several models in nature and became a branch of Mathematics with a growing number of applications.

In Physics, probability theory became an important calculus tool at the same time as Thermodynamics and, later, Quantum Physics.

It has been ascertained that determinist phenomena have a very small part in surrounding nature.

The vast majority of phenomena from nature and society are stochastic (random).

Their study cannot be deterministic, and that is why hazard science was raised as a necessity.

There are almost no scientific fields in which probability theory is not applied. Also, sociology uses the calculus of probabilities as a principal tool.

Moreover, some commercial domains are based on probabilities (insurance, bets, and casinos, among others).

Probability as a limit

We initially present the probability concept as a limit of a sequence of real numbers. Although this is a particular result (a theorem, namely *The Law of Large Numbers*) and not a definition, it confers to probability a structure on whose base the comprehension of the concept becomes clearer and more accessible to average people.

Also, the perception of probability as a limit diminishes the risk of error of qualitative interpretation with regard to the real behavior of random phenomena, given the mathematical model.

Although in the strictly mathematical chapter the chronology of presentation of the notions and results is the natural one from a scientific point of view (definitions – axioms – theorems), here we deliberately reverse this order by explaining the probability concept in a particular case, as a limit, for a sequence of independent experiments that aim at the occurrence of a certain event.

This presentation mode is chosen with a didactic purpose because the notion of *limit* is easier to explain, visualize and assimilate at a nonacademic level than the complete definition of probability based on Kolmogorov's axioms. This last one is explained for beginners' understanding in the next section.

In addition, in mathematics, the status of *definition* and *theorem* may commute within the same theory, without affecting the logic of the deduction process. This means the deduced property of a mathematical object can stand for its definition itself, and vice versa.

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The relative frequency

In the section titled *Probability – the word*, we defined the probability of event A occurring as being the ratio between the number of tests favorable for A to occur and the number of all equally possible tests.

Let us come back to the classical example experiment of rolling the die. Let A be the event *occurrence of number 5*.

According to the classical definition, the probability of A is $P(A)=\frac{1}{6}$ (one favorable case; namely, the die shows the side with the number 5 from six equally possible cases). By establishing the probability $P(A)$, we have attached a unique number to event A (1/6 in this case).

The ratio from the definition of probability is positive and less than 1, so probability is a function P defined on the set of events generated by an experiment, with values in interval [0, 1]. This function attaches a positive and less than 1 number to each event and this number is called an event's probability. The properties of this function, as well as its extension to sets of more complex events, is exposed in the section titled *Probability as a measure.*

The simple assignment of a number to a singular event through function P does not provide additional information about the occurrence of respective event. The fact that we know the probability of occurrence of number 5 at die rolling as being 1/6 does not confer any relevance to the prediction that this event will happen or not at a given moment as result of a singular test.

But there exists a mathematical result, called the Law of Large Numbers, which gives us additional information about the occurrence of an event within a sequence of experiments.

This information is about the frequency of occurrence of the event within the sequence of tests and is a property of limit making the connection between relative frequency and probability.

Let us see what relative frequency means.

Staying with the example experiment of rolling the die and taking the same event A (occurrence of number 5), let us assume we have a sequence of independent experiments (tests) $E_1, E_2, E_3, ..., E_n, ...$, each generating a certain outcome.

We can choose this sequence as being the chronological sequence of tests of rolling the die, performed by same person over time, or the chronological sequence of such tests performed by whatever number of established persons. We can also choose the sequence as being the chronological sequence of such tests performed by all the people on Earth who make up this experiment.

No matter the set of chosen tests, as long as they are well defined and form a sequence (an infinite enumeration can be attached to them). Obviously, any of these choices is hypothetical.

Within this sequence of independent tests $(E_n)_{n\geq 1}$, we define the relative frequency of occurrence (producing) of event A.

Let us assume the die rolls a 2 on the first test (E_1). At this moment (after one test), the number of occurrences of 5 is 0.

Denote: $E_1 - 1 - 0$

On the second test (E_2), the die rolls a 1. At this moment (after two tests), the number of occurrences of 5 is still 0.

Denote: $E_2 - 2 - 0$

On the third test (E_3), the die rolls a 4. At this moment (after three tests), the number of occurrences of 5 is still 0.

Denote: $E_3 - 3 - 0$

On the fourth test (E_4), the die rolls a 5. At this moment (after four tests), the number of occurrences of 5 is 1.

Denote: $E_4 - 4 - 1$

On the fifth test (E_5), the die rolls a 2. At this moment (after five tests), the number of occurrences of 5 is still 1.

Denote: $E_5 - 5 - 1$

And so on, assume we obtain the following results:

$$E_1 - 1 - 0$$
$$E_2 - 2 - 0$$
$$E_3 - 3 - 0$$
$$E_4 - 4 - 1$$
$$E_5 - 5 - 1$$
$$E_6 - 6 - 1$$
$$E_7 - 7 - 1$$
$$E_8 - 8 - 1$$
$$E_9 - 9 - 2$$
$$E_{10} - 10 - 2$$
$$\ldots\ldots\ldots\ldots$$
$$E_n - n - a_n$$
$$\ldots\ldots\ldots\ldots$$

In the previous diagram, the first column contains the successive tests, the second column contains the order numbers within the sequence and the third column contains the cumulative numbers of occurrences of 5. The successive results from the third column (the total number of occurrences of 5 after each test) form a sequence $(a_n)_{n\geq 1}$ (observe in the diagram that they are values of a function defined on *N*; namely, an infinite enumeration). Thus, we have obtained the sequence 0, 0, 0, 1, 1, 1, 1, 1, 2, 2,…, a_n, … of cumulative numbers of occurrences of event *A* as the result of the performed tests.

Probability theory does not provide us with any property of this sequence $(a_n)_{n\geq 1}$, so we have no information about certain of its terms, nor about its behavior to infinity. What we can only observe is the sequence being monotonic increasing (see the section titled *Sequences of real numbers. Limit* in the mathematical chapter). In exchange, the theory provides information about another sequence, namely the sequence $\left(\frac{a_n}{n}\right)_{n\geq 1}$.

By making the ratio a_n / n for each test, we can complete the previous diagram with a new column containing the numerical values of these ratios:

$$
\begin{array}{l}
E_1 - 1 - 0 - 0 \\
E_2 - 2 - 0 - 0 \\
E_3 - 3 - 0 - 0 \\
E_4 - 4 - 1 - 1/4 \\
E_5 - 5 - 1 - 1/5 \\
E_6 - 6 - 1 - 1/6 \\
E_7 - 7 - 1 - 1/7 \\
E_8 - 8 - 1 - 1/8 \\
E_9 - 9 - 2 - 2/9 \\
E_{10} - 10 - 2 - 2/10\ (= 1/5) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
E_n - n - a_n - a_n / n \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

Obviously, the results from the last column form a sequence, as being values of a function defined on N (0, 0, 0, 1/4, 1/5, 1/6, 1/7, 1/8, 2/9, 2/10,…, a_n / n, …).

The general term of this sequence is $\frac{a_n}{n}$ and is called the *relative frequency* of occurrence of event A.

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Probability as a measure

In the previous section we presented probability as a number associated with an event generated by an experiment with a finite number of outcomes.

The probability was defined particularly, on a finite set of events, in which the elementary events are equally possible.

The notions of elementary event, field of events and equally possible events were not defined rigorously, but only through examples.

In this context, the probability of an event is defined as the ratio between the number of cases that are favorable for the event to occur and the total number of equally possible cases.

By following this definition, we can easily calculate the probabilities of events generated by experiments like rolling the die, spinning the roulette wheel, picking a card from a given deck, drawing an object from an urn whose content is known, etc.

We can apply the classical definition in these cases because each of the experiments enumerated generates a finite number of events and the elementary events are equally possible.

The classical definition of probability does not refer to the set of events associated with an experiment, which may be more or less complex, but only to its number of elements.

Once this set is organized (structured) as a field with certain axioms (properties), the probability can be defined as a function whose properties are generated by the structure of the respective field and can be studied in greater depth.

In mathematics, a structure is assumed to organize a set of objects (which could be numbers, points, sets, functions, and the like) by introducing a group of axioms enunciating properties of the elements with respect to certain relations of operations (composition laws) that are well defined for that set.

By granting a set a well-defined structure, the set acquires the denomination of field, space, corpus, algebra, and so on.

In theory, organizing sets as structures is generally done to study the properties of certain functions that are defined on those sets.

Here are few examples of structures:

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In the algebra $\mathcal{P}(\Omega)$, we have $V = \Omega$ and $\Lambda = \phi$.

We also have a definition of minimal elements with respect to the relation order $\subset$ (the equivalent of the implication relation $\Rightarrow$ from logics), called *atoms:*

An element $\mathcal{A}$ of a Boole algebra $\mathcal{A}$, $A \neq V$ is called atom of that algebra, if the inclusion $B \subset A$ implies $B = \Lambda$ or $B = A$, for any $B \in \mathcal{A}$.

In the algebra $\mathcal{P}(\Omega)$, each part of Ω having one single element is an atom of this algebra.

The set Σ of the events associated with an experiment, along with previously defined operations between events, form a Boole algebra. This result can be immediately deduced if it is taken into account that the events can be represented as sets and Σ is included in $\mathcal{P}(\Omega)$, but it also can be stated as an axiom if, from a rigorousness excess, we do not identify the event with the set of tests that generate it.

Between the specific notions of a Boole algebra and those of the set of events associated with an experiment, we can observe the following correspondence:

Boole algebra	**The set of events (Σ)**
operation $\cup$	operation *or*
operation $\cap$	operation *and*
operation ^{C}	operation *non*
null element Λ	impossible event (ϕ)
total element V	sure event (Ω)
atom	elementary event

The Boole algebra of the events associated with an experiment is called the *field of events* of the respective experiment.

A field of events is then a set of results Ω, structured with an algebra of events Σ and is denoted by $\{\Omega, \Sigma\}$.

Thus, we have attached an algebraic structure to each experiment; namely, the Boole algebra of the set of events Σ. This action creates the basis of the mathematical model on which the real phenomenon can be studied, making possible the step from the practical experiment to probability theory.

Granting the set Σ with an algebraic structure has the goal of conferring consistency to the ulterior definition of probability as a function on Σ and as well of providing us the tools needed to deduce the properties of this function. These properties stand for the basic formulas of applied probability calculus.

The first step in extending the classical definition of probability is defining it as a function of a finite field of events.

A field of events $\{\Omega, \Sigma\}$ is finite if the total set Ω is finite.

The next definition calls probability a function on a finite field of events, which has three certain properties.

Let $\{\Omega, \Sigma\}$ be a finite field of events.

Definition: Call probability on Σ, a function P: $\Sigma \rightarrow R$ meeting the following conditions:

(1) $P(A) \geq 0$, for any $A \in \Sigma$;

(2) $P(\Omega) = 1$;

(3) $P(A_1 \cup A_2) = P(A_1) + P(A_2)$, for any $A_1, A_2 \in \Sigma$ with $A_1 \cap A_2 = \phi$.

From this definition, it follows that:

1) A probability takes only positive values;

2) The probability of the sure event is 1;

3) The probability of a compound event consisting of two incompatible events is the sum of the probabilities of those two events.

Probability is defined then as a function P on the field of events associated with an experiment, which meets the three conditions (axioms) described above.

The fact that $\{\Omega, \Sigma\}$ is a field of events (that reverts to the fact that Σ is structured as a Boole algebra) ensures:

– The membership $\Omega \in \Sigma$ (therefore the expression $P(\Omega)$ from condition (2) does make sense);

– The commutativity of the operation of union (*or*) between events and the membership $A_1 \cap A_2 \in \Sigma$ (therefore the expression $P(A_1 \cup A_2)$ from condition (3) does make sense).

This information ensures the total consistency of the definition.

Property (3) can be generalized by recurrence for any finite number of mutually exclusive events. Therefore, if $A_i \cap A_j = \phi$, $i \neq j$, $i, j = 1, \ldots, n$, then:

$P(A_1 \cup A_2 \cup \ \ldots \ \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$ or, else written, $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$.

The axioms of a Boole algebra (commutativity and associativity) are also involved here, allowing the above denotations by defining in a solid way to operate a *finite union* within an algebra of events.

This is the definition of probability in a finite field of events. Let us name it *Definition 1*. It does not identify a unique function, called probability, but it attaches this name to a function defined in the field of events, which has the three specific properties.

It is then possible for several probabilities (as functions) to exist in the same field of events, all having the same properties presented in the mathematical chapter.

Definition 1 does make sense as long as the set of events Σ has a Boolean structure (it is a field of events). Therefore, we cannot talk about the probability of an isolated, singular, predefined event. We can do that only in the context of its membership in a field of events.

Thus, probability makes sense only as a function of a well-defined set that is structured as a field of events.

A finite field of events $\{\Omega, \Sigma\}$, along with a probability *P*, is called a *finite probability field* and is denoted by $\{\Omega, \Sigma, P\}$.

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As a whole, the following ideas must be kept in mind:

- Probability is nothing more than a measure; as length measures distance and area measures surface, probability measures aleatory events. As a measure, probability is in fact a function with certain properties, defined on the field of events generated by an experiment.
- A probability is characterized not only by the specific function *P*, but by the entire aggregate *the set of possible outcomes of the experiment – the field of generated events – function P,* called probability field; probability makes no sense and cannot be calculated unless we initially rigorously define the probability field in which we operate.

● Probability is not a punctual numerical value; textually given an event, we cannot calculate its probability without including it in a more complex field of events. Probability as a number is in fact a limit, respectively the limit of the sequence of relative frequencies of occurrences of the event to measure, within a sequence of independent experiments.

The understanding of probability is complete when it hints not only at the concept's definition, but also at the relationship between the mathematical model and the real world of random processes.

In this direction, the next sections of this chapter are useful for completing the general image created thus far by defining probability.

Relativity of probability

When we speak about the relativity of probability, we refer to the real objective way in which probability theory models the hazard and in which the human degree of belief in the occurrence of various events is sufficiently theoretically justified to make decisions.

Thus, any criticism of the application of probability results in daily life will not hint at the mathematical theory itself, but at the transfer of theoretical information from the model to the surrounding reality.

Probability theory was born from humans' natural tendency to predict happenings and the unknown. The starting point was the basic experimental observation of the behavior of relative frequencies of occurrence of events within the same type of experiment: according to long-time experience, the relative frequency of occurrence of a certain event oscillates around a certain value, approximating it with high enough accuracy after a very large number of tests. Aiming to demonstrate this result, probability theory was created step by step and integrated with measure theory, and the experimentally observed property of relative frequency was theoretically proved and called the Law of Large Numbers.

The hazard

Apparently paradoxically, as probability theory developed, the terms *hazard* and *randomness* disappeared from its language, even though they represented the reality object that initially stimulated its creation.

The mathematical model created for probability, which started from hazard and randomness, is exclusively dedicated to defining the measure–function that reproduces the probability behavior of events within a structure, as well as to the deduction of the properties of this function and its statistical applications.

Randomness, as the object of objective reality, is not defined or introduced in the mathematical context of theory. At most, the introductions to the various papers in this domain only make reference to it.

Although the philosophy of hazard has always stood for an attractive field, we note that no one from among the great philosophers has studied the hazard as a philosophical object.

In Kantian language, the hazard does not stand for the thinker as a logical category, nor as an *a priori* form, nor even as a precise experimental category, remaining at its simple status of a word that covers unclassified circumstantial situations of the various theories.

Besides philosophy, mathematics did not succeed in providing a rigorous definition; moreover, it did not create a solid model for randomness and hazard.

Emile Borel stated that, unlike other objects from the surrounding reality for which the creation of models assumes an idealization that preserves their properties, this idealization is not possible in the case of hazard. In particular, whatever the definition of a sequence formed by the symbols 0 and 1 is, this sequence will never have all the properties of a sequence created *at random*, except if it is experimentally obtained (for example, by tossing a coin in succession and putting down 0 if the coin shows heads and 1 if the coin shows tails). Borel also proposes an inductive demonstration scheme for this affirmation regarding the *random sequence*.

Assume someone is building such an indefinite sequence 0010111001…, which has all properties of the sequences generated by experimental randomness. Assume the first n terms of the sequence were built and follows to write the $n + 1$ term. There are two options: the first n results are somehow taken into account or they are not. The first option annuls the random character of the construction, because a precise rule for choosing that term exists. The second option brings us to the same situation we would stand in at the beginning of the sequence's construction: How do we choose one of the symbols 0 or 1 without taking into account any difference between these? This choice would be equivalent to a draw, which assumes an experimental intervention. Using the reduction to the absurd method, we come to the conclusion that such a random sequence cannot be built.

Far from entirely convincing us, this proof underlines the theoretical difficulties of conception with regard to this subject.

Borel asserts that we cannot state a constructive definition of a random sequence, but we can attach to these words an axiomatic–descriptive definition. However, he also admits that such a definition would have no mathematical effectiveness with respect to its integration into probability theory.

In the attempt to mathematize randomness, Richard von Mises has tried to enunciate an axiomatic definition by introducing the notion of *collective*. He defines a sequence of elements $a_1, a_2, ..., a_n, ...$ as a random sequence (and calls it *collective*) if, given a property f of its elements and denoting by $n(f)$ the number of elements from the first n having the property f, the following two axioms are met:

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These degrees of belief are, in fact, the expression of the relative frequency: in a long succession of rolls, the number 3 will occur one-sixth of the time.

This gambling behavior is amenable, with the reservation of relativity of application of mathematical model, to a regular play that generates a long succession of experiments, but it has nothing to do with a theoretical motivation in the case of an isolated bet.

And yet, most gamblers have this expectation and decision-making behavior, which is nothing more than a subjective translation of the conclusions of the Law of Large Numbers.

This example can be generalized for any type of bet for which a decision is based on relative frequencies, as well for any situation in which the subjective degree of belief is superimposed on mathematical probability: the theoretical result is transposed in a finite context (although infinity cannot be obtained at a practical experience level), conferring to the application a relative character.

Transferring the limit–probability concept in a practical situation and transforming it into a degree of belief represents at least an act of relative judgment.

Similarly, the same relativity supervenes in a case in which we consider probability as a measure and we apply its properties in practical cases of finite experiences.

The mathematical concept of measure–probability has been defined in an infinite collective, and at a practical level, infinity cannot be reproduced experimentally.

Even excluding the applicability of probability, the measure notion itself has a relative quantitative character that results from the comparative measurement (existence of a standard); the result of a measurement does not represent an absolute numerical value, but a multiple or division of the standard of measure used.

As the length of a segment measured in meters represents the number (integer or fraction) of inclusions of meter in each respective segment, the probability of an event represents the fraction corresponding to that event, as part of a sure event (with the measure 1). That the probability of occurrence of the number 3 on a die roll is 1/6 is translated into comparative measurement terms as event *occurrence of 3* weights 1/6 from the sure event.

Still, in the classical definition of probability (the ratio between the number of favorable cases for the event to occur and the number of all equally possible cases) a relativity is involved, even from the construction of the definition. This definition applies only to fields of events in which the elementary events are equally possible.

This attribute of *equally possible* can be textually defined as *probable is the same size* or *possible in the same measure,* which are terms that revert by their content to the term *probability* and make this notion defined through itself.

In addition, applying this definition in practical situations assumes the idealization of *the elementary events are equally possible,* which confers to probability another relativity, resulting from the nontotal equivalence of the mathematical model with the real phenomenon.

The approximation *all the die's numbers have the same probability of occurrence,* which is indispensable in probability calculus applied to more complex events related to this experiment, is questionable at a practical level and raises the question whether ignoring all physical factors equally is justified.

Assuming a die is a perfect cube, the mode of rolling can alter the aleatory attribute of the experiment.

For example, repeated throws at a certain angle might favor the occurrence of a certain die's sides, as would the height from which the die is thrown, the initial impulse or the way it is held.

The unconditioned ignoring of all these physical factors comes from a certain symmetry of objects and physical actions and assumes an approximation that confers to the practical probability calculus another relativity.

Obviously, viewing the experimental results from the perspective of relative frequency in long successions of experiments (in the case of rolling a die, this operation might be performed over time by several persons, at various angles and from various heights), the relativity previously mentioned dissipates and the Law of Large Numbers finally takes effect, even if delayed.

But the *equally possible* act of approximation still remains a necessary idealization and probability theory could not be built without it.

Still related to applicability, a major relativity of probability exists that may significantly change the results of practical calculus. It is about choosing the probability field in which the application runs.

As a mathematical object, the probability field is a triple $\{\Omega, \Sigma, P\}$, where Ω is the set of possible results attached to an experiment, Σ is a field of events on Ω, and P is a probability on Σ. If any of these three components is changed, the result is a new mathematical object, respectively, a new probability field.

This means we can obtain different probabilities for the same event, even if it is included in different fields of events.

We take again a simple example presented earlier in this chapter as an application of the classical definition of probability.

A 52-card deck is shuffled and the first card from the upper side is face up. Let us calculate the probability for the card shown to be clubs (♣). The experiment has 52 possible outcomes (results), from which thirteen are favorable for the event *the first card is clubs* to happen. The probability of this event is then 13/52.

Let us now change the conditions of the experiment by assuming that we accidentally saw the last card after the deck was shuffled, and this was 5♦. The event *the first card is clubs* now has the probability 13/51 because the number of favorable cases is still thirteen, but the number of all possible results is 51 because the last card (5♦) cannot be the first.

Therefore, for the same event *E* that was textually described as *the first card is clubs,* we found two different probabilities.

This double calculation is far from being a paradox, but it has a very simple mathematical explanation:

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The Philosophy of Probability

What is the sense of the question: "What is the probability of …"? This seems to be the essential question around which all problems of philosophy revolve.

Great mathematicians like Pascal, Bernoulli, Laplace, Cornot, von Mises, Poincaré, Reichenbach, Popper, de Finetti, Carnap and Onicescu performed philosophical studies of the probability concept and dedicated to them an important part of their research, but the major questions still remain open to study:

- Can probability also be defined in other terms besides through itself?
- Can we verify that it exists, at least in principle? What sense must be assigned to this existence? Does it express anything besides a lack of knowledge?
- Can a probability be assigned to an aleatory isolated event or just to some collective structures?

These are just few of the basic questions that philosophy dealt with, through the efforts of the thinkers listed earlier, but still without a scientifically satisfactory conclusion.

Hundreds of pages of papers might be written on each of such kind of questions. We do not pretend in this section that we review all the problems of philosophy of probability, nor do we claim that the text was optimally organized from a didactic point of view. This presentation aims only to stimulate the research and deep knowledge tendencies of readers with regard to this subject, to complete an image of the probability concept that includes its philosophical aspects and to extend the simple image of a mathematical tool of calculus of degree of belief, which is so common among average people.

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What does verisimilar actually mean in reality? From the history of analysis of this concept, and further from antiquity we have the famous example of Carneade:

A boy goes into a dark cellar. He sees there a rope that looks like a coiled snake. The boy gets scared, not knowing whether the rope is a snake or not. He waits a little and observes that the object does not move. Maybe it is not a snake. He carefully advances a few steps and the object still remains motionless. His belief that it is a lifeless object increases. He takes a stick and touches it. The object does not move. Only when he takes it in his hand does he reach the practical certitude.

This example contains in itself the subjective probability promoted by probability scientists such as de Finetti. The example shows very intuitively the evolution from uncertainty to practical certainty via Bayes's theorem.

Carneade's rivals raised the following question to him:

Verisimilitude is something resembling the truth. If you do not know what the truth is, how can you know that something looks like it?

These polemical requests made Carneade develop a nuanced analysis of the concept of verisimilitude. He considered that the degree of belief attached to things is given by:

– Vivacity of sensations
– Order of representations
– Absence of internal contradictions.

In the rope example, at the beginning the vivacity of sensations was not enough (it was dark there), then the boy appealed to the order of representations by verifying whether the rope was moving, and the certitude came as result of finding no internal contradictions between rope's representation and the object in front of him.

If a mouse under the rope had made it move, the boy's level of belief in the hypothesis *it is a rope* would have diminished owing to the internal contradictions between representations: a rope does not move by itself.

Carneade also says: "We, humans, cannot reach the absolute truth through reason, nor perception, as skeptics proved. The truth, the certitude are not given to us. Still, life requires us to act. Therefore, we have to be content with the practical appearance of truth—the verisimilitude."

……………….. missing part …………………………

The Psychology of Probability

From the previous sections dedicated to relativity and philosophy of probability, we realize that this concept has a major psychological component generated by its impact with the human mind at the cognitive level.

The probability notion itself, through its interpretation as a degree of belief, as well as the application of probability theory in daily life, are the subject of objective or subjective human appreciations and judgments, no matter the human's level of knowledge.

The human mind is built so that it manifests through two apparently contradictory tendencies: on the one hand, it is eager for knowledge and disposed to the mental effort of searching for answers to questions about phenomena from the surrounding world.

On the other hand, it accepts the comfort of immediate explanations and theories that do not contradict other convictions, at least at first.

In this sense, the interpretation of probability as an absolute or at least a sufficient degree of belief (to make decisions) has a partially solid motivation: thus far, probability theory is the only valid theory operating upon aleatory events (even if in idealized context) through incontestable mathematical methods.

This motivation is in fact the expression of the comfort we just talked about: humans search around them for explanations, causes and theories to answer to their questions conveniently and chooses the most rigorous (but not always).

They limit their mental effort to this plane search (on the horizontal), by omitting (intentionally or not) that another dimension of cognition (on the vertical) exists where profound study is done through abstraction and generalization.

Obviously, this last type of cognition process assumes judgments that are not at everybody's hand, a certain level of education, intelligence and conceptual perception being required.

Another factor that influences the thinking process is the subconscious, this copilot of the human mind that much of the time, without our being aware, takes control over the functions of the organism, including the cognitive ones.

The two distinct notions—philosophical probability and mathematical probability—even when individually perceived, are frequently confused by a person in situations involving theoretical judgment or applications.

Besides this major conceptual inconsistency, qualitative interpretation errors may be multiple:

● The exclusive use of the term *probability* in the sense of its classical definition;

Not every field of events can be reduced to a finite field with equally possible elementary events, for the classical definition to apply.

● Attaching a probability to an isolated event;

The field of events as a domain of the probability function must be structured as a Boole algebra. The probability of an event does not make sense if that event does not belong to such a field.

● Isolation of probability as a function from the probability field;

Probability is determined by the triple (the set of possible outcomes–the field of events–the probability function); namely, the probability field. Defining the probability of an event means putting in evidence each of the three components, not just the function.

● Identification of probability with the relative frequency;

Although probability is the limit of relative frequency and prediction can be made only for a long-running succession of experiments, the result is frequently applied by analogy to an arbitrary succession of experiments and even to an isolated event.

● Transformation of mathematical probability into an absolute degree of belief;

Neglecting the relative aspects of probability turns it into an absolute criterion in making decisions in various situations that require an action.

These interpretation errors are related to the intellectual capacity as well as to the intimate psychological mechanisms of the subject, including the functions of the subconscious.

Among them, the most important interpretation error from a psychological point of view is the transformation of probability into an absolute degree of belief, which also has social implications, because it results immediately in the acceptance of probability as a unique decision-making criterion with effects in the sphere of personal actions that also affect other people around the subject.

Including this psychical behavior in the category of qualitative interpretation error finds its complete motivation in the section titled *Relativity of probability*, where we saw that the probability notion has a multirelative character, with respect to both the concept itself and to its equivalence relation with the phenomenal world it models.

Making an optimal recommendation about this line of decision-making conduct is a complex problem, which itself represents another individual theory.

It is easy to make a recommendation at an immediate classification level: changing the attribute of *absolute* into *relative*.

The subject must mentally perceive the (mathematical) probability as a rigorously calculated degree of belief, but *relative* with respect to the possibility of the physical occurrence of an event.

This attribute assumes in succession the acceptance of probability as a decisional criterion, but not exclusively. The decision come from a complex of criteria, perhaps weighted, some of which may be even subjective.

Mathematics is still firmly involved in sustaining such recommendation and also in an eventual theory of subjective probability.

De Finetti declared himself with no doubt as sustaining a subjectivist concept of probability: “My point of view may be considered as the extreme of subjectivistic solutions. The purpose is to show how the logical laws of probability theory can be rigorously established from a subjective point of view; among other things, it will be shown how, although I refuse to admit the existence of any objective value and signification of probabilities, we can make an idea about the subjective reasons due to which, in a big part of different problems, the subjective judgments of normal people are not only not much different, but they rigorously coincide with each other."

He obsessively repeats in his papers that "Probability does not exist," being completed by Barlow with "...except in our mind."

De Finetti proposes an extremely simple definition for probability: "Assume a person is constrained to evaluate the ratio p at which he or she would be disposed to change an amount S, which may be positive or negative, depending on the occurrence of an event E, with the sure possession of amount pS. Then, p is said to be the probability of E given by the respective person."

This manner of defining probability is, obviously, disputable.

First, the definition makes no sense if referring to an empirical subject, for the simple reason that ratio p does not actually depend only on event E, but also on the amount S, as psychological research has revealed: if a normal person feels indifferent about the alternative *$1 for sure against $6 if a 6 occurs at a die roll*, an alternative of the type *$10000 for sure against $60000 if a 6 occurs at a die roll* would not be felt so indifferently by the same person, who would prefer the first variant.

But just the hypothesis $p = p(E)$ and not $p = p(E, S)$ is essential for de Finetti's axiomatization (p is a function of E and not also of S).

This can be also observed in a demonstration of the additivity of p, which is rigorous only if p does not depend on S.

At this objection, the author answers: "It would have been better if I would deal with utilities, but I was aware of the difficulties of bets and I preferred to avoid them by considering small enough stakes. Another lack of my definition—or, better stated, of the tool I choose to make it operational—is the possibility for those accepting the bet against the respective person to have better information. This would lead us in game theory situations. Of course, any tool is imperfect and we have to be content with an idealization ...

Probability theory is not an attempt to describe the real behavior of people, but refers to coherent behavior and the fact that people are more or less coherent is not essential."

But let us admit that we can skip the drawback of initial definition by postulating that this is the way an ideal person would act (the ratio p to only depend on E), a person who is not interested in winning, but uses the bet only to clarify his or her own initial subjective probabilities that persist in the subconscious.

Once this deadlock is passed, de Finetti develops an elegant and coherent theory about which he says that it succeeded in formalizing the probability concept that is the closest to the one used by common people, the one used by people in their practical judgments.

The found rules "are in fact only the precise expression of the rules of logics of probable, which are unconsciously, qualitatively, even numerically applied by all people, in any life situation."

As a conjecture of conclusions on his theory, de Finetti puts the following question: "Among the infinity of possible estimations, is there one we could consider as objectively coherent, in an undefined sense for the moment? Or, can we say at least about an evaluation that is better than another? ... For assigning an objective sense to probability notion, during the centuries two schemes were imagined: the scheme of equally possible cases and the frequency considerations. But none of these procedures obliges us to admit the existence of an objective probability. On the contrary, if someone wants to force their significance for reaching such conclusion, will face the well-known difficulties, which are vanishing themselves if we become less pretentious, that is if we will try not to eliminate, but to define the subjective element existing between them ... The problem consists in considering the coincidence of opinions as a psychological fact; the reasons of that fact can retain their subjective character, which cannot be left aside without raising a lot of problems whose sense is not even clear."

Regarding the decisional criteria based on a degree of belief, it has been ascertained that a human resorts psychologically to practical statistics in large measure.

As in the case of using mathematical probability as an absolute decision-making criterion, consulting the statistical results often becomes the unique criterion for making a decision in certain situations.

For example, a person may decide to undergo a certain operation if the medical statistics reveal a satisfactory success rate (let us say, 80 percent) for this type of operation.

Such a decision is based exclusively on the results of previous practical statistics, which the undecided person deems trustworthy.

Many times decisions are also made on the basis of previous unrecorded statistics whose results are established through personal observations and intuition.

For example, when we observe black clouds in the sky, we hurry to get home. The statistics used here as the basis of a decision are the set of personal observations performed during a lifetime of the same phenomenon: in most situations (p% from them, p enough big), it rained when such clouds were in the sky. Then, it is very possible (p% degree of belief) that it will rain again now, so we quicken our steps to get home sooner.

Such decisions are not, in practice, the immediate result of consulting the statistics, but are a process that transforms the statistical result into a degree of belief, which is still an expression of a probability.

Probability has a direct theoretical connection with statistics (mathematical statistics is an extension of probability theory and is even considered a part of it), but is also connected to practical statistics.

Practical statistics means a collection of outcomes (results) of a certain type of experiment, recorded over time.

These results correspond to a finite time period (that explains the use of the term *statistics until this moment)*, and therefore to a finite number of experiments.

Generally, solid practical statistics cover a very large time period and, by implication, a very large number of experiments.

The statistical results are recorded in diagrams or bidimensional tables, in which time (experiment's number) is one of coordinates.

Such representation shows the relative frequency of the event or events that are the subject of statistics, even though this frequency is not directly calculated (as a ratio), with the condition that the experiments are performed in identical conditions.

From here, the connection with probability is immediate. Being a relative frequency, the more results the statistical record contains, the better this frequency approximates the probability of a respective event.

Thus, the degree of belief of the person making the decision based on statistics becomes a translation of a partial result (the relative frequency recorded by statistics until that moment) of an isolated event (respective decision-making situation).

The subject, in fact enlarges the succession of statistically recorded experiments with another one, the one he or she is involved in that has not yet happened, by applying the relative frequency corresponding to the previous n experiments to the $n + 1$ experiments including the virtual $n + 1$ experiment.

This in inductive reasoning resembling prediction based on frequential probability.

In the previous medical example, the patient makes the following judgment: If the operation succeeded in 80 percent of previous cases, it will also succeed in my case with 80 percent certainty, so I will have the operation and assume the 20 percent risk.

In reality, this risk might be lower or higher because of other criteria not taken into account. The 80 percent certainty does not come from objective factors specific to the isolated medical case of that patient, but from practical statistics, which correspond to other persons.

……………….. missing part …………………………

Here is a simple example of false intuition, in which the error comes from an incorrect framing of the problem:

You have the following information: A person has two children, at least one a boy. What is the probability of the other child being a boy, too?

You are tempted to answer 1/2, by thinking there are only two choices: boy and girl.

In fact, the probability is 1/3, because the possible situations are three:

boy–girl (BF)
girl–boy (FB)
boy–boy (BB)

and one of them is favorable, namely (BB).

The initial information refers to both children as a group and not to a particular child from the group.

In the case of estimation of 1/2 , the error comes from establishing the sample space (and, implicitly the field of events) as being {B, F}, when in, fact it, is a set of ordered pairs: {BF, FB, BB}.

The probability would be 1/2 if one of the two children had been fixed by hypothesis (as example: *the oldest is a boy* or *the tallest is a boy).*

For many people, the famous birthday problem is another example of contradiction with their own intuitions:

If you randomly choose twenty-four persons, what do you think of the probability of two or more of them having the same birthday (this means the same month and the same day of the year)?

Even if you cannot mentally estimate a figure, intuitively you feel that it is very low (if you do not know the real figure in advance).

Still, the probability is 27/50, which is a little bit higher than 50 percent!

A simple method of calculus to use here is the step by step one:

The probability for the birthday of two arbitrary persons not to be the same is 364/365 (because we have one single chance from 365 for the birthday of the first person to match the birthday of the second).

The probability for the birthday of a third person to be different from those of the other two is 363/365; for the birthday of a fourth person is 362/365, and so on, until we get to the last person, the 24th, with a 342/365 probability.

We have obtained twenty-three fractions, which all must be multiplied to get the probability of all twenty-four birthdays to be different. The product is a fraction that remains as 23/50 after reduction.

The probability we are looking for is the probability of the contrary event, and this is 1 – 23/50 = 27/50.

This calculus does not take February 29 into account, or that birthdays have a tendency to concentrate higher in certain months rather than in others. The first circumstance diminishes the probability, while the second increases it.

If you bet on the coincidence of birthdays of twenty-four persons, on average you would loose twenty-three and win twenty-seven of each fifty bets over time.

Of course, the more persons considered, the higher the probability.

With over sixty persons, probability gets very close to certitude.

For 100 persons, the chance of a bet on a coincidence is about 3000000 : 1.

Obviously, absolute certitude can be achieved only with 366 persons or more.

One of the most curious behaviors based on false intuition is that of lottery players, where the winning probabilities are extremely low.

As a game of chance that offers the lowest winning odds, it is not predisposed to strategies. The player (regular or not) purely and simply tries his or her fortune, whether he or she knows the involved mathematical probabilities beforehand.

Still, too few players stop contributing to the lottery, even when they hear or find what the real probability figures are.

In a *6 from 49* lottery system, the probability of winning the 1st category with a single played variant (six numbers) is 1/13983816.

If playing weekly during a lifetime (let us assume eighty years of playing, respectively, 4320 draws), the probability for that player to finally win improves to 1/3236.

Still assuming that the person plays ten or even 100 variants once, he or she has a probability of 1/323 or 1/32, which is still low for a lifetime. And we did not even take into account the amount invested.

What exactly makes lottery players persevere in playing by ignoring these figures?

Beyond the addiction problems, there is also a psychological motivation of reference to community, having observation as a unique criterion.

A regular player may ask himself or herself the question: "If people all around me win the lottery, why can't I have my day once, too?"

Probability theory cannot completely answer to that question, but in exchange it can answer the question why that player has not won until the present moment: because the probability of winning is very close to zero.

Another example of false intuition is still related to the lottery.

Most players avoid playing the variant 1, 2, 3, 4, 5, 6. Their argument is intuitive: It is impossible for the first six numbers to be drawn.

Indeed, it is almost impossible, in the sense that the probability of drawing that variant is 1/13983816.

Still, this probability remains the same for any played variant (assuming the technical procedure of drawing is absolutely random).

There are no preferential combinations, so that particular variant has not at all an inferior status from point of view of possibility of occurrence.

Moreover, if someone won by playing that variant, the amount won would be much higher than in the case of other played variant, because the winning fund will be divided (eventually) among fewer players.

Thus, the optimal decision would be to play that particular variant instead of others. Of course, this decision remains optimal as long as most players are not acquainted with this information.

False intuition successfully manifests in several gaming situations in gambling.

The so-called feeling of the player, which at a certain moments will influence a gaming decision, is very often a simple illusory psychical reaction that is not analytically grounded.

Probability represents one of the domains in which intuition may play bad tricks, even for persons with some mathematical education.

Therefore, intuition must not be used as a calculus tool or for probability estimation.

A correct probability calculus must be based on minimal, but clear, mathematical knowledge and must follow the basic logical algorithm of the application process, starting with framing the problem, then establishing the probability field and the calculus itself.

This process is described in detail in chapter titled *Beginner's Calculus Guide.*

PROBABILITY THEORY BASICS

In this chapter we present the main set of notions and foundation results for the mathematical concept of probability and probability theory.

Because this guide is addressed principally to beginners, we have limited it to the notions leading to the rigorous definition of probability and the properties generating the formulas that are necessary to practical calculus as well, especially for discrete and finite cases.

Denotation convention

In this chapter as well as in the following chapters that contain solved applications, the entire range of denotations corresponding to a specific operation or definition are used without their being limited.

For example, for the operation of multiplication, we use the symbols $"\times", "\cdot"$ or no sign (in case of algebraic products that contain letters); for the operation of division we use the symbols $":", "-"$ or $"/"$; and for convergence we use the denotations $"\lim_{n\to\infty} a_n = a", "a_n \xrightarrow{n\to\infty} a", "a_n \xrightarrow{n} a"$, etc.

Fundamental notions

Sets

The set concept is a primary one in the sense that it cannot be defined through other mathematical notions.

In mathematics, the word *set* represents any well-defined collection of objects of any type (in the sense that we can decide whether a certain object belongs to respective collection or not) called elements of the set.

Specifying a set means enumerating its constituent objects or indicating a specific property of these objects (a common property that other objects do not have).

The sets are denoted in uppercase letters and the description of their elements is enclosed in braces.

Examples: $A = \{x, y, z\}$; $B = \{2, 3, 5, 8\}$; $C = \{x \in R, 1 \le x \le 3\}$.

In certain theoretical constructions, we refer to objects that belong to a certain class of elements as a base set or reference set, usually denoted by Ω.

Example: The set of real valued numbers R can be considered a base set for its subsets: the set of integer numbers Z, of natural numbers N, of positive real numbers R_+.

A set can contain a finite or infinite number of elements. A set with no elements is called an *empty set* and is denoted by ϕ.

Example: The set of natural satellites of the Moon is an empty set.

If A is a set, B is called a *subset* of A if any element of B is also an element of A.

If Ω is a base set, we denote by $\mathcal{P}(\Omega)$ the *set of all parts* of Ω.

So $\mathcal{P}(\Omega)$ contains a number of parts A, B, C, … that are individually well defined as sets. Therefore, the elements of $\mathcal{P}(\Omega)$ are subsets of Ω.

Two sets are said to be *equal* if they contain the same elements.

Consider the following operations on $\mathcal{P}(\Omega)$:

1. The *union* of two sets A and B, denoted by $A \cup B$, is the set of elements that belong to A or B.

Example: If $A = \{2, 3, 5, 7\}$ and $B = \{3, 5, 9, 11\}$, then $A \cup B = \{2, 3, 5, 7, 9, 11\}$.

2. The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements that belong to both A and B.

In the previous example, $A \cap B = \{3, 5\}$.

If $A \cap B = \phi$, we call the sets A and B *mutually exclusive* or *disjoint*.

3. The *complement* of A, denoted by A^C, is the set of elements from Ω that do not belong to A. Obviously, $A \cup A^C = \Omega$.

4. The *difference* of sets A and B, denoted by $A - B$ or A / B, is the set of elements from A that do not belong to B.

In the above example, $A - B = \{2, 7\}$ and $B - A = \{9, 11\}$.

Say that set A is *included* in set B (it is a subset of B) and denote it by $A \subset B$ (or $B \supset A$), if any element from A is an element of B.

The following properties, which are intuitively obvious, are characteristic for $\mathcal{P}(\Omega)$:

1. $\Omega \in \mathcal{P}(\Omega)$, $\phi \in \mathcal{P}(\Omega)$.
2. $A \in \mathcal{P}(\Omega) \Rightarrow A^C \in \mathcal{P}(\Omega)$.
3. $A, B \in \mathcal{P}(\Omega) \Rightarrow A \cup B \in \mathcal{P}(\Omega)$.
4. $A, B \in \mathcal{P}(\Omega) \Rightarrow A \cap B \in \mathcal{P}(\Omega)$.

……………….. missing part …………………………

Field of events. Probability

Probability theory deals with the laws of evolution of random phenomena. Here are some examples of random phenomena:

1. The simplest example is the experiment involving rolling a die; the result of this experiment is the number that appears on the upper side of the die. Even though we repeat the experiment several times, we cannot predict which value each roll will take because it depends on many random elements like the initial impulse of the die, the die's position at the start, characteristics of the surface on which the die is rolled, and so on.
2. A person walks from home to his or her workplace each day. The time it takes to walk that distance is not constant, but varies because of random elements (traffic, meteorological conditions, and the like).
3. We cannot predict the percentage of misfires when firing a weapon a certain number of times at a target.
4. We cannot know in advance what numbers will be drawn in a lottery.

In these experiments, the essential conditions of each experiment are unchanged. All variations are caused by secondary elements that influence the result of the experiment.

Among the many elements that occur in the phenomena studied here, we focus only on those that are decisive and ignore the influence of secondary elements. This method is typical in the study of physical and mechanical phenomena as well as in technical applications.

In the study of these phenomena, there is a difference of principle between the methods that allow the essential elements that determine the main character of the phenomenon to be taken into account and those methods that do not ignore the secondary elements that lead to errors and perturbations.

The randomness and complexity of causes require special methods of study of random phenomena, and these methods are elaborated by probability theory.

The application of mathematics in the study of random phenomena is based on the fact that, by repeating an experiment many times in identical conditions, the relative frequency of a certain result (the ratio between number of experiments having one particular result and total number of experiments) is about the same, and oscillates around a constant number. If this happens, we can associate a number with each event; that is, the probability of that event. This link between structure (the structure of a field of events) and number is the equivalent of the mathematics of the transfer of quality into quantity.

The problem of converting a field of events into a number is equivalent to defining a numeric function on this structure, which has to be a measure of the possibility of an event occurring. Because the occurrence of an event is probable, this function is named *probability.*

Probability theory can only be applied to phenomena that have a certain stability of the relative frequencies around probability (homogeneous mass phenomena). This is the basis of the relationship between probability theory and the real world and daily practice.

So, the scientific definition of probability must first reflect the real evolution of a phenomenon.

Probability is not the expression of the subjective level of man's trust in the occurrence of the event, but the objective characterization of the relationship between conditions and events, or between cause and effect.

The probability of an event makes sense as long as the set of conditions is left unchanged; any change in these conditions changes the probability and, consequently, the statistical laws governing the phenomenon.

The discovery of these statistical laws resulted from a long process of abstraction. Any statistical law is characterized, on the one hand, by the relative inconstancy or the variability of various objects' activity, and therefore we cannot predict the evolution of an individual object. On the other hand, for a large set of phenomena a stable constancy takes place, and this can be expressed by the statistical law.

Practical statistics works first with finite fields of events, while physical and technical experiments take place on infinite fields of events.

Field of events

In probability theory, the studied events result from random *experiments* (trials) and each performance of an experiment is called a *test.*

The result of a test is called an *outcome.* An experiment can have more than one outcome, but any test has a single outcome.

An *event* is a set of outcomes.

As a result of a test with outcome *e*, an event *A occurs* if $e \in A$ and *does not occur* if $e \notin A$.

Example:

In the roll of a die, the set of all possible outcomes is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Some events are: $A = \{1, 3, 5\}$ – uneven number, $B = \{1, 2, 3, 4\}$ – less than the number 5, $C = \{2, 4, 6\}$ – even number.

If rolling a 3, events *A* and *B* occur.

Denote by Ω the set of all possible outcomes of an experiment and by $\mathcal{P}(\Omega)$ the set of all parts of Ω.

Ω is called the set of outcomes or the *sample space*.

The random events are elements of $\mathcal{P}(\Omega)$.

On the set Σ of the events associated with an experiment, we can introduce three operations that correspond to the logical operations *or, and, non.* Let $A, B \in \Sigma$.

a) *A or B* is the event that occurs if, and only if, one of the events *A* or *B* occurs. This event is denoted by $A \cup B$ and is called the union of events *A* and *B*.

b) *A and B* is the event that occurs if, and only if, both events *A* and *B* occur. This event is denoted by $A \cap B$ and is called the intersection of events *A* and *B*.

c) *non A* is the event that occurs if, and only if, event A does not occur. This event is called the complement (opposite) of A and is denoted by A^C.

If we attach to each event the set of tests through which it occurs, then the operations between events revert to the respective operations between sets of corresponding tests, so the designations a), b) and c) are justified.

The results of the operations with events are also events attached to their respective experiments.

If $A \cap B = \phi$, meaning A and B cannot occur simultaneously, we say that A and B are *incompatible* (mutually exclusive) events.

If $A \cup B = \Omega$, we say that A and B are *collectively exhaustive*.

In the set Σ of events associated with a certain experiment, two events with special significance exist, namely, event $\Omega = A \cup A^C$ and event $\phi = A \cap A^C$.

The first consists of the occurrence of event A or the occurrence of event A^C, which obviously always happens; that means this event does not depend on event A. It is natural to call Ω the *sure event.*

Event ϕ consists of the occurrence of event A and the occurrence of event A^C, which can never happen. This event is called the *impossible event.*

Let $A, B \in \Sigma$. We say that event A implies event B and write $A \subset B$, if, when A occurs, B necessarily occurs.

If we have $A \subset B$ and $B \subset A$, we say that events A and B are equivalent and write $A = B$ (this reverts to the equality of the sets of tests that correspond to respective events).

The implication between events is a partial order relation on the set of events and corresponds to the inclusion relation from Boole algebras.

<u>Definition</u>: An event $A \in \Sigma$ is said to be *compound* if two events $B, C \in \Sigma$, $B \neq A$, $C \neq A$ exist, such that $A = B \cup C$. Otherwise, the event A is said to be *elementary*.

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Probability properties

We have the following properties of the probability function:

(P1) For any $A \in \Sigma$, we have $P(A^C) = 1 - P(A)$.

(P2) $P(\phi) = 0$.

(P3) Any $A \in \Sigma$, $0 \le P(A) \le 1$.

(P4) Any $A_1, A_2 \in \Sigma$ with $A_1 \subset A_2$, we have $P(A_1) \le P(A_2)$.

(P5) Any $A_1, A_2 \in \Sigma$, we have $P(A_2 - A_1) = P(A_2) - P(A_1 \cap A_2)$.

(P6) If $A_1, A_2 \in \Sigma$, $A_1 \subset A_2$, then $P(A_2 - A_1) = P(A_2) - P(A_1)$

(P7) Any $A_1, A_2 \in \Sigma$, we have $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.

(P8) Any $A_1, A_2 \in \Sigma$, we have $P(A_1 \cup A_2) \le P(A_1) + P(A_2)$.

(P9) If $(A_i)_{1 \le i \le n} \subset \Sigma$, then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{j<i} P(A_i \cap A_j) +$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

This property is also called the *inclusion-exclusion principle*.

(P10) Let $(A_i)_{1 \le i \le n} \subset \Sigma$ be events, with $P(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \ne 0$. Then:

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) P(A_2 / A_1) P(A_3 / A_1 \cap A_2) \ldots P(A_n / A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

The previous properties represent formulas currently used in probability calculus on a finite field of events.

Property (P9) is the main calculus formula for applications in finite cases.

Applications:

1) Two shooters are simultaneously shooting one shot each at a target. The probabilities of target hitting are 0.8 for the first shooter and 0.6 for the second. Calculate the probability for the target to be hit by at least one shooter.

Answer:

Let A_i – *shooter number i* ($i = 1, 2$) *hits the target* be the events.

The events A_1 and A_2 are independent, so $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$ (See the section titled *Independent events. Conditional probability*). We have, according to (P7):

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) =$$
$$= 0.8 + 0.6 - 0.8 \times 0.6 = 0.92.$$

2) A batch of 100 products is amenable to quality control. The batch will be rejected if at least one defective product among five randomly controlled products is found. Assuming the batch contains 4 percent defective products, calculate the probability for the batch to be rejected.

Answer:

Denoting by A the event *the batch must be rejected,* we calculate $P(A^C)$.

We denote by A_k the event *controlled product number k is accepted (not defective),* $1 \le k \le 5$. Events A_k are not independent.

We have:

$$P(A^C) = P(A_1 \cap ... \cap A_5) = P(A_1)P(A_2 / A_1)...$$

$$P(A_5 / A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{96}{100} \cdot \frac{95}{99} \cdot \frac{94}{98} \cdot \frac{93}{97} \cdot \frac{92}{96} \text{ and}$$

$P(A) = 1 - P(A^C)$. (We have used (P10) and (P1))

3) An urn contains twenty balls numbered from 1 to 20. If we choose four different numbers between 1 and 20 and randomly extract four balls from urn, we can calculate the probability for at least two extracted balls to have numbers from the chosen four.

Answer (to follow this solution, read the chapter titled *Combinatorics*):

We call a *variant* any group of four different extracted balls. The whole number of possible variants is $C_{20}^4 = 4845$. Without restricting the generality, we can denote the chosen numbers by 1, 2, 3, 4 (the probability is the same for any other four numbers).

A variant containing numbers 1 and 2 has a (1, 2, x, y) form, with x, y receiving different values ($x \neq y$) from 20 – 2 = 18 numbers.

The number of combinations taken two at a time from the chosen four numbers is $C_4^2 = 6$, namely, (1, 2), (1, 3), (1, 4), (2, 3), (2,4), (3, 4). Let:

A_1 – the extracted variant contains the balls numbered 1 and 2,

A_2 – the extracted variant contains the balls numbered 1 and 3,

A_3 – the extracted variant contains the balls numbered 1 and 4,

A_4 – the extracted variant contains the balls numbered 2 and 3,

A_5 – the extracted variant contains the balls numbered 2 and 4,

A_6 – the extracted variant contains the balls numbered 3 and 4.

We can write, as sets:

$A_1 = \{(1, 2, x, y); x \neq y;\ x, y \in \{1, 2, ..., 20\} - \{1, 2\}\}$,

$A_2 = \{(1, 3, x, y); x \neq y;\ x, y \in \{1, 2, ..., 20\} - \{1, 3\}\}$,

$A_3 = \{(1, 4, x, y); x \neq y;\ x, y \in \{1, 2, ..., 20\} - \{1, 4\}\}$,

$A_4 = \{(2, 3, x, y); x \neq y;\ x, y \in \{1, 2, ..., 20\} - \{2, 3\}\}$,

$A_5 = \{(2, 4, x, y); x \neq y;\ x, y \in \{1, 2, ..., 20\} - \{2, 4\}\}$,

$A_6 = \{(3, 4, x, y); x \neq y;\ x, y \in \{1, 2, ..., 20\} - \{3, 4\}\}$.

Each of sets A_i ($i = 1, \ldots, 6$) has $C_{18}^2 = 153$ elements (within a variant, the order of numbers does not matter).

We must calculate $P(A_1 \cup A_2 \cup \ldots \cup A_6)$. To do this, we apply property (P9) (the inclusion-exclusion principle).

.... missing part ....

Observe that the two conditions from the definition of probability imply the axioms in the definition of measure.

Therefore, this probability is a measure with $\mu(\Omega) = 1$, so it acquires all the properties of a measure.

The terms currently used in the measure theory and those used in probability theory correspond as follows:

Measure Theory	**Probability Theory**
Measurable space	Field of events
Tribe	σ-field of events
Measurable set	Event
Whole space	Sure event
Empty set	Impossible event
Measure space	Probability field
Measure of a set	Probability of an event

The properties of probability on a finite field also stand for the probability σ-fields (the properties (P1) to (P10) from the section titled *Probability on a finite field of events*).

In addition, if $\{\Omega, \Sigma, P\}$ is a probability σ-field, we also have the following properties:

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The law of large numbers

The previous theoretical statements defined the concept of probability on a field of events and offered tools for basic probability calculus.

The result we present now is qualitative; in fact, it illustrates the way in which probability models the hazard.

We enunciate the Law of Large Numbers, not in its general mathematical form, in order to avoid having to define more complex concepts, but in an exemplified particular form, in a way that everyone can understand.

The particular enunciation is the following classic result, known as Bernoulli's Theorem:

The relative frequency of the occurrence of a certain event in a sequence of independent experiments performed under identical conditions converges toward the probability of that event.

The theorem states that if A is an event, (E_n) a sequence of independent experiments, a_n the number of occurrences of event A after the first n experiments, then the sequence of non-negative numbers $\left(\frac{a_n}{n}\right)$ is convergent and its limit is $P(A)$:

$$a_n / n \xrightarrow{n\to\infty} P(A)$$

The expression a_n is called *frequency* and the expression $\frac{a_n}{n}$ is called *relative frequency*. We exemplify this expression by considering the classical experiment of tossing a coin:

………………… missing part …………………

Classical discrete probability repartitions

Bernoulli scheme

Let us consider the following problem:

Three independent shots are fired at a target.

The probability of hitting the target is p for each of the three shots. Find the probability for two of the three shots hitting the target.

Let A be the event *two of the three shots hit the target* and let A_i $(i = 1, 2, 3)$ be the events *shot number i hit the target.*

A can be written as:

$$A = \left(A_1 \cap A_2 \cap A_3^C\right) \cup \left(A_1 \cap A_2^C \cap A_3\right) \cup \left(A_1^C \cap A_2 \cap A_3\right).$$

The three parentheses are incompatible and the events they consist of are independent. This gives us the result:

$$P(A) = P(A_1)P(A_2)P(A_3^C) + P(A_1)P(A_2^C)P(A_3) + P(A_1^C)P(A_2)P(A_3)$$

Therefore, $P(A) = 3p^2(1-p)$.

We can solve the following more general problem using a similar approach:

Consider that n independent experiments are performed. In each experiment, event A may occur with probability p and does not occur with probability $q = 1 - p$. Find the probability for event A to occur exactly m times in the n experiments.

Let B_m be the event *A occurs exactly m times in the n experiments* and let A_i $(i = 1, 2, ..., n)$ be the events *A did not occur in the i-th experiment.*

Each variant of occurrence of B_m consists of m occurrences of event A and of $n - m$ nonoccurrences of A (that is $n - m$ occurrences of A^C).

We then have:

$$B_m = \left(A_1 \cap A_2 \cap ... \cap A_m \cap A_{m+1}^C \cap ... \cap A_n^C\right) \cup$$
$$\left(A_1^C \cap A_2 \cap ... A_{m+1} \cap A_{m+2}^C \cap ... \cap A_n^C\right) \cup ... \cup$$
$$\left(A_1^C \cap ... \cap A_{n-m}^C \cap A_{n-m+1} \cap ... \cap A_n\right).$$

The number of ways we can choose m experiments in which A occurs from the n experiments is C_n^m.

All variants are incompatible and the experiments are independent, so:

$$P(B_m) = P_{m,n} = \underbrace{p^m q^{n-m} + \ldots + p^m q^{n-m}}_{C_n^m} = C_n^m p^m q^{n-m}.$$

Probabilities $P_{m,n}$ have the form of the terms from the development of the binomial $(p+q)^n$.

This is why the field from this scheme (repartition, distribution) is called the binomial field (its elementary events can be considered elements of the Cartesian product $\Omega^n = \Omega \times \cdots \times \Omega$).

J. Bernoulli, especially, made this probability scheme the subject of research, and that is why it is also called *Bernoulli scheme.*

The mean and dispersion of a random variable X that is binomially obtained can be easily calculated.

They are $M(X) = np$ and $D^2(X) = npq$.

Example:

Two fighters with equal strength box 12 rounds (the probability for any of them to win a round is 1/2). Calculate the mean, dispersion and standard deviation of the random variable representing the number of rounds won by one fighter.

Answer:

The random variable X has the binomial repartition:

$P(X = k) = C_{12}^k \left(\frac{1}{2}\right)^{12}$, $k = 1, \ldots, 12$. We have $M(X) = 6$, $D^2(X) = 3$ and $D(X) = \sqrt{3}$.

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COMBINATORICS

Combinatorial analysis plays a major role in probability applications, from a calculus perspective, because many situations deal with permutations, combinations or arrangements.

The correct approach to combinatorial problems and the ease of handling combinatorial calculus are 50 percent of the probability calculus abilities for games of chance.

Therefore, this chapter contains many solved and unsolved practical applications that are useful for learning this calculus.

As in the previous chapter on mathematics, the theoretical discussions present only definitions and important results, without demonstrations.

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In detail, the minimal numerical calculus for the two formulas is represented by the following algorithm:

a) For arrangements A_n^m:

1) Calculate the difference $n - m$.
2) Calculate the product of all consecutive numbers from $n - m + 1$ to n.

Examples:

– Let us calculate A_7^3:

1) $7 - 3 = 4$
2) Calculate the product of all numbers from 4 + 1 = 5 to 7; namely, $5 \cdot 6 \cdot 7 = 210$.

$A_7^3 = 210$

– Let us calculate A_{10}^5:

1) $10 - 5 = 5$

2) Calculate the product of all numbers from 5 + 1 = 6 to 10; namely, $6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 42 \cdot 720 = 3024$.

$A_{10}^{5} = 3024$

Other examples:

$A_{8}^{3} = 6 \cdot 7 \cdot 8 = 42 \cdot 8 = 336$

$A_{17}^{7} = 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 = 98017920$

b) For combinations C_{n}^{m}:

1) Calculate the difference $n - m$.

2) Calculate the product of all consecutive numbers from $n - m + 1$ to n.

3) Calculate the product of all consecutive numbers from 1 to m.

4) Divide the result obtained in step 2 by the result obtained in step 3.

Examples:

– Let us calculate C_{5}^{2}:

1) $5 - 2 = 3$

2) Calculate the product of all numbers from 3 + 1 = 4 to 5; namely, $4 \cdot 5 = 20$.

3) Calculate the product of numbers from 1 to 2; namely, $1 \cdot 2 = 2$.

4) Divide the result obtained in step 2 by the result obtained in step 3, namely $20 : 2 = 10$.

$C_{5}^{2} = 10$

– Let us calculate C_{12}^{5}:

1) $12 - 5 = 7$

2) Calculate the product of all numbers from 7 + 1 = 8 to 12; namely, $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12$.

3) Calculate the product of all numbers from 1 to 5; namely, $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$.

4) Divide the result from step 2 by the result from step 3, by writing it as the fraction $\frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$; after the immediate reductions, we get $8 \cdot 9 \cdot 11 = 792$.

$C_{12}^{5} = 792$

Leaving the products from steps 2 and 3 in their uncalculated factorial form is recommended, because this allows the reduction of the fraction from step 4, and this operation spares us of a lot of additional calculations.

Other examples:

$$C_{15}^{3} = \frac{13 \cdot 14 \cdot 15}{1 \cdot 2 \cdot 3} = 13 \cdot 7 \cdot 5 = 13 \cdot 35 = 455$$

$$C_{19}^{4} = \frac{16 \cdot 17 \cdot 18 \cdot 19}{1 \cdot 2 \cdot 3 \cdot 4} = 4 \cdot 17 \cdot 3 \cdot 19 = 12 \cdot 17 \cdot 19 = 3876$$

The property $C_n^m = C_n^{n-m}$ is useful in calculations, in the sense of simplifying them. This property of combinations must be used when the difference *n* – *m* is less than *m*, because it reduces the number of product factors.

For example, for *n* = 57 and *m* = 53, we have $C_{57}^{53} = C_{57}^{4}$.

Obviously, C_{57}^{4} is more easily developed and calculated by applying the formula than C_{57}^{53} because it contains the factorial 4! (instead of 53!): $C_{57}^{4} = \frac{54 \cdot 55 \cdot 56 \cdot 57}{4!}$, $C_{57}^{53} = \frac{5 \cdot 6 \cdot ... \cdot 57}{53!}$.

The property $C_n^m = C_n^{n-m}$ must be applied, in general, when *m* and *n* are close in value to one another.

Examples:

$$C_7^5 = C_7^2 = \frac{6 \cdot 7}{2} = 21$$

$$C_{11}^{8} = C_{11}^{3} = \frac{9 \cdot 10 \cdot 11}{2 \cdot 3} = 3 \cdot 5 \cdot 11 = 165$$

$$C_{25}^{19} = C_{25}^{6} = \frac{20 \cdot 21 \cdot 22 \cdot 23 \cdot 24 \cdot 25}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 10 \cdot 7 \cdot 22 \cdot 23 \cdot 5 = 177100$$

At the final step of the calculus algorithm, after we obtain the last fraction, we can check for possible errors by following both the numerator and the denominator, which must show all the consecutive factors of the products from previous steps.

Because the number of combinations (arrangements, permutations) is natural, total reduction of that fraction (complete evanescence of the denominator) is obligatory.

If we find that the fraction cannot be reduced in totality, it is a sign that an error has occurred at the previous steps, and that calculus must be redone.

Partitioning combinations

Let us do the following exercise: unfold all 4-size combinations from the numbers (1, 2, 3, 4, 5, 6, 7). The whole number of these must be $C_7^4 = C_7^3 = 35$.

This unfoldment must not be aleatory because doing otherwise risks losing combinations; in addition, some combinations end up being written more than once. To make a proper unfoldment, we use the following algorithm: While permanently maintaining the ascending order of the numbers from written combinations (from left to right), we successively increase the numbers starting from the right. When increasing the numbers is no longer possible, we increase the number from the first previous place of combination and continue the procedure until the capacity to increase is no longer possible.

Choosing the ascending order does not affect the overall approach to solving the problem because the order of elements does not count within a combination. This procedure ensures that all combinations are enumerated, without omission, and avoids any repetition (double counting).

Here is how this unfoldment works concretely:

Start with the combination (1234). Successively increase number 4 from the last (fourth) place: (1235), (1236), (1237). Increasing the number from the last place is no longer possible. Increase number 3 from the third place by replacing it with 4: (1245), (1246), (1247). Follows to replace 4 with 5 on the third place: (1256), (1257).

Replace 5 with 6 on the third place: (1267). Follows to change the second place: replace 2 with 3 and start with 4 on the third place: (1345), (1346), (1347).

Replace 4 with 5 on the third place: (1356), (1357).

Replace 5 with 6 on the third place: (1367). Again follows to change the second place with 4 instead of 3: (1456), (1457).

Replace 5 with 6 on the third place: (1467).

Replace 4 with 5 on the second place: (1567).

Now, the combinations having number 1 on the first place are exhausted. Replace 1 with 2 on the first place and start with 3 on the second place and 4 on the third place: (2345), (2346), (2347).

Put 5 on the third place: (2356), (2357).

Put 6 on the third place: (2367).

Replace 3 with 4 on the second place, and start with 5 on the third place: (2456), (2457).

Put 6 on the third place: (2467).

Replace 4 with 5 on the second place: (2567). Now, the combinations having number 2 on the first place are exhausted. Put 3 on the first place and start with 4 on the second place and 5 on the third place: (3456), (3457).

Put 6 on the third place: (3467).

Replace 4 with 5 on the second place: (3567). Now, the combinations having number 3 on the first place are also exhausted. Replace 3 with 4 on the first place: (4567). This is the last counted combination.

By counting all these unfolded combinations, we find thirty-five different combinations; therefore, the unfoldment is correct.

We performed this algorithmic unfoldment in order to grasp the effective process of generating the combinations of elements of a given set and to see why repeated multiplications are the only elementary operations involved in the process of counting.

In the example above, observe that:

– For counting the 4-size combinations containing number 1, we fix this number and unfold all 3-size combinations containing the rest of numbers (2, 3, 4, 5, 6,7), which are in number of $C_6^3 = 20$.

Indeed, by effectively counting the unfolded combinations containing 1, we find twenty combinations.

A combination containing the number 1 can be written as (1*xyz*), where *x*, *y* and *z* are mutually different variables with values in the set $\{2, 3, 4, 5, 6, 7\}$.

By an abuse of denotation, we can write (1*xyz*) = (1)(*xyz*) = $1 \cdot C_6^3 = 20$, this means: the number of 4-size combinations containing the number 1 is equal to number of 1-size combinations containing 1 (namely, a single combination) multiplied by the number of 3-size combinations from the set of numbers left (2, 3, 4, 5, 6, 7).

Obviously, the same result (twenty) also stands for the number of combinations containing any of the numbers 2, 3, 4, 5, 6 or 7.

Note that the total number of 4-size combinations of elements from the given set is not equal to the sum of these partial results (number of combinations containing 1 + number of combinations containing 2 + … + number of combinations containing 7)!

This can be verified immediately ($35 \neq 20 \cdot 6$) and is explained by the fact that, through addition, some combinations are counted more than once (a combination containing 1 could also contain 2, etc.).

– For counting the 4-size combinations containing the numbers 1 and 2, we fix the numbers 1 and 2 and unfold all 2-size combinations of elements from the set of numbers left (3, 4, 5, 6, 7), in number of $C_5^2 = 10$.

By the same abuse of denotation, we can write (12*xy*) = (12)(*xy*) = $1 \cdot C_5^2 = 10$: the number of 4-size combinations containing the numbers 1 and 2 is equal to number of 2-size combinations containing 1 and 2 (namely, a single combination) multiplied by the number of 2-size combinations from the set of numbers left (3, 4, 5, 6, 7). Obviously, the same result (ten) also stands for the number of combinations containing any two given numbers (23), (35), (57), etc.

– For counting the 4-size combinations containing the numbers 1, 2 and 3, we fix the numbers 1, 2 and 3 and unfold all 1-size combinations of elements from the set of numbers left (4, 5, 6, 7), in number of $C_4^1 = 4$.

By the same abuse of denotation, we can write (123x) = (123)(x) $= 1 \cdot 4 = 4$: the number of 4-size combinations containing the numbers 1, 2 and 3 is equal to number of 3-size combinations containing 1, 2 and 3 (namely, a single combination) multiplied by the number of 1-size combinations from the set of numbers left (3, 4, 5, 6, 7). Obviously, the same result (four) also stands for the number of combinations containing any three given numbers (234), (357), (136), etc.

In the previous exercise, the count of combinations containing given numbers having the form (1xyz), (12xy) or (123x) stands for classic examples of problems that impose additional conditions on the elements of combinations.

Their solution uses the graphic representations (1xyz) = (1)(xyz), (12xy) = (12)(xy) and (123x) = (123)(x), which simplify the calculus by reductions to lower size combinations.

Call this procedure *partitioning* of combinations.

The immediate generalization of this exercise is as follows:

Let $E = A \cup B \cup C \cup D$ be a set, with A, B, C and D being mutually exclusive.

If from all 4-size combinations of elements of E we want to count those containing one element from A, we write:

($abcd$) = (a)(bcd), where $a \in A$ (this is the additional condition imposed on the elements of combinations); (a) is the number of elements of A and (bcd) the number of 3-size combinations of elements from $B \cup C \cup D$; these two numbers are multiplied; observe that the sets A and $B \cup C \cup D$ are disjoint.

If we want to count the combinations containing two elements from $A \cup B$, we write:

($abcd$) = (ab)(cd), where $a, b \in A \cup B$; (ab) is the number of 2-size combinations of elements of $A \cup B$ and (cd) the number of 2-size combinations of elements of $C \cup D$; observe that the sets $A \cup B$ and $C \cup D$ are exclusive (this is an obligatory condition for the partitioning procedure to be applied; otherwise, some combinations would be counted more than once).

The procedure stands for combinations of any size, regardless of the conditions on their elements. The generalization is as follows:

Let us consider the n-size combinations $\left(a_1 a_2 ... a_n\right)$ of elements of a finite set A. Let $\left(k_1, k_2, ..., k_m\right)$ be a partition of n (k_i are natural numbers such that $k_1 + k_2 + ... + k_m = n$) and A_{k_1}, A_{k_2}, ..., A_{k_m} a partition of the set A (A_{k_i} are mutually exclusive sets such that $\bigcup_{i=1}^{m} A_{k_i} = A$). Then:

$$\left(a_1 a_2 ... a_n\right) = \left(a_1 a_2 ... a_{k_1}\right)\left(a_{k_1+1} ... a_{k_1+k_2}\right) ... \left(a_{k_1+...+k_{m-1}+1} ... a_n\right)$$

As we stated earlier, this is an abuse of denotation and represents a procedure rather than a formula. The partitioning of combinations is, in fact, a graphic representation that allows us to view a property and simplify the calculus.

Graphic representations, and literal denotations, even those that are considered abusive, are highly recommended in combinatorial problems. They aid the correct framing of a problem by applying the proper properties and the correct performance of calculus.

The following examples show how these procedures work in concrete applications.

Applications

Solved applications

1) We have fifteen books and we must fill a shelf that can accommodate only eleven books.

a) How many ways can we arrange eleven books in the shelf, by choosing from the fifteen?

b) How many ways can we choose the eleven books?

Solution:

a) We can directly apply the arrangement formula; the searched number is A_{15}^{11}; according to the minimal calculus algorithm for arrangements, we find:

1) 15 – 11 = 4

2) Do the product of numbers from 4 + 1 = 5 to 15:

$5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 =$
$30 \cdot 56 \cdot 90 \cdot 132 \cdot 13 \cdot 14 \cdot 15 = 54486432000$.

$A_{15}^{11} = 54486432000$

b) The choices do not take order into account.

We deal here only with combinations, and the searched number is C_{15}^{11}.

We may use the property $C_{15}^{11} = C_{15}^{15-11} = C_{15}^{4}$ to reduce the calculus and then follow the minimal calculus algorithm for combinations:

1) $15 - 4 = 11$
2) Do the product of numbers from 12 to 15: $12 \cdot 13 \cdot 14 \cdot 15$
3) Do the product of numbers from 1 to 4: $2 \cdot 3 \cdot 4$
4) Do the ratio $\dfrac{12 \cdot 13 \cdot 14 \cdot 15}{2 \cdot 3 \cdot 4}$; after reductions, we get

$13 \cdot 7 \cdot 15 = 1365$.

$C_{15}^{11} = 1365$

2) How many ways can we arrange the letters of the word MAJORITY?

Solution:

The letters are different and total seven, so the total number of permutations is $7! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 24 \cdot 30 \cdot 7 = 24 \cdot 210 = 5040$.

3) At 6/49 lottery system (six numbers are drawn from a total of forty-nine, from 1 to 49, with one played variant having six numbers), calculate the total number of possible variants that can be drawn.

Solution:

A drawn variant represents a 6-size combination from forty-nine numbers, so the searched number is C_{49}^{6}:

1) $49 - 6 = 43$
2) Do the product $44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49$
3) Do the product $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$

4) Do the ratio

$$\frac{44\cdot 45\cdot 46\cdot 47\cdot 48\cdot 49}{2\cdot 3\cdot 4\cdot 5\cdot 6} = 44\cdot 3\cdot 46\cdot 47\cdot 48\cdot 49 = 13983816 = C_{49}^{6}.$$

(Let us hope that gamblers who usually play using a few variants are not too disappointed!)

4) In a 6/49 lottery system, calculate how many possible variants containing the numbers 5 and 11 exist.

Solution:

By fixing the two numbers, the variants containing them will have the form (5 11 *xyzt*), where *x*, *y*, *z* and *t* are distinct numbers belonging to the set $\{1, 2, ..., 49\} - \{5, 11\}$, which has 49 – 2 = 47 elements. We do the partitioning (5 11 *xyzt*) = (5 11)(*xyzt*).

(5 11) represents one combination and the number of (*xyzt*) combinations is given by the number of 4-size sets that can be built from the numbers left (47), namely, C_{47}^{4}. Then, the searched number is $1\cdot C_{47}^{4} = C_{47}^{4} = \frac{44\cdot 45\cdot 46\cdot 47}{2\cdot 3\cdot 4} = 178365$.

5) In a 6/49 lottery system, calculate how many variants containing the numbers 1, 2 and 3 exist.

Solution:

The respective variants will have a (123*xyz*) form, with *x*, *y* and *z* mutually different and different from 1, 2 and 3. The set from which *x*, *y* and *z* can take values has 46 elements (49 – 3).

By partitioning, we have (123*xyz*) = (123)(*xyz*).

Denoting by *C* the searched number, we have:

$C = 1\cdot C_{46}^{3} = C_{46}^{3} = \frac{44\cdot 45\cdot 46}{2\cdot 3} = 15180$ (1 is the number of (123) combinations and C_{46}^{3} is the number of (*xyz*) combinations).

6) In a 6/49 lottery system, calculate: a) how many variants containing only even numbers exist; b) how many variants containing only uneven numbers exist.

Solution:

a) The even numbers are 2, 4, 6, 8, …, 48, and number 24 in total (we have $2=1\cdot 2$, $4=2\cdot 2$, $6=3\cdot 2$, …, $48=24\cdot 2$; the count was done by following the first factor of the products).

The number of combinations of six even numbers is then

$$C_{24}^{6}=\frac{19\cdot 20\cdot 21\cdot 22\cdot 23\cdot 24}{2\cdot 3\cdot 4\cdot 5\cdot 6}=134596.$$

b) The uneven numbers are 1, 3, 5, 7, …, 49, and total 49–24=25 (we have subtracted the number of even numbers).

The number of combinations of six uneven numbers is then

$$C_{25}^{6}=\frac{20\cdot 21\cdot 22\cdot 23\cdot 24\cdot 25}{2\cdot 3\cdot 4\cdot 5\cdot 6}=177100.$$

7) In a 6/49 lottery system, find the number of possible variants containing exactly four even numbers.

Solution:

Let us denote such variant by (*PPPPxy*), where *P* are less than forty-nine even numbers and x, y are distinct and uneven.

This is obviously an abuse of denotation. Repeating the same letter *P* does not mean those numbers are equal (a combination cannot contain identical elements); it only means they are even and mutually different. This denotation simplifies the partitioning and calculus by showing the elements on which the additional conditions were imposed.

(*PPPPxy*) = (*PPPP*)(*xy*)

The number of (*PPPP*) combinations is the number of 4-size combinations from twenty-four (the number of even numbers); namely, C_{24}^{4} , and the number of (*xy*) combinations is the number of 2-size combinations from twenty-five (the rest of numbers; namely, the uneven ones), respectively C_{25}^{2}.

We then calculate:

$$C_{24}^{4}\cdot C_{25}^{2}=\frac{21\cdot 22\cdot 23\cdot 24}{2\cdot 3\cdot 4}\cdot\frac{24\cdot 25}{2}=3187800.$$

8) In a 5/40 lottery system (six numbers are drawn from a total of forty, from 1 to 40, with one played variant having five numbers), calculate the total number of possible variants that can be drawn.

.... missing part ....

17) In Texas Hold'em, how many ways can the pocket cards be dealt to a player so that they contain at least one ace?

Solution:

Denoting an ace by *A* (*A* can be A♣, A♠, A♦, A♥), we can count the combinations of (*Ax*) form, with $x \neq A$ and (*AA*) form, the results following to be added.

For (*Ax*): *A* takes four values and *x* takes 52 – 4 = 48 values, so the number of combinations is $4 \cdot 48 = 192$.

For (*AA*): The number of combinations is $C_4^2 = 6$.

Then, the number of combinations containing at least one ace is 192 + 6 = 198.

18) In Texas Hold'em, how many ways can the pocket cards be dealt to a player so that they to contain two identical symbols?

Solution:

For a specific symbol *S*, the combinations to be counted have an (*SS*) form and their number is $C_{13}^2 = \dfrac{12 \cdot 13}{2} = 78$. We have four symbols, so the total number of combinations is $4 \cdot 78 = 312$.

19) You are participating in a Texas Hold'em game and are dealt the 7♣ and the 8♦. How many ways can the flop cards be distributed so that they contain: a) exactly two clubs; b) a minimum of two clubs?

Solution:

a) A flop combination containing exactly two clubs has a (*CCx*) form, with $x \neq C$ (*C* = clubs); *C* takes 13 – 1 = 12 values (a clubs card from a total of 13 is in the hand); the set of values *x* can take has 52 – 2 – 2 – 10 = 38 elements (we have subtracted the two pocket cards, the two *CC* cards from the flop combination and the ten clubs left). (*CCx*)=(*CC*)(*x*)

The number of combinations is $C_{12}^2 \cdot 38 = \dfrac{11 \cdot 12}{2} \cdot 38 = 2508$.

b) Let us calculate the number of combinations containing three clubs. These have a (*CCC*) form and number $C_{12}^{3} = \frac{10 \cdot 11 \cdot 12}{2 \cdot 3} = 220$.

In total, we have 220 + 2508 = 2728 combinations containing at least two clubs.

20) In Texas Hold'em, how many possible hands containing a pair (any) can a player be dealt?

Solution:

For a specific pair (*PP*) (*PP* are two different cards of same value), there are $C_4^2 = 6$ possible combinations (*P* takes four values). We have thirteen values for all cards; therefore, the total number of combinations is $13 \cdot C_4^2 = 13 \cdot 6 = 78$.

21) At a slot machine (a machine with 3, 4 or 5 reels, each reel have the same number of different symbols, with 1, 2 or 3 winning lines; after an aleatory spin of the reels, a combination of symbols that may win or not occurs on the winning line) with three reels, seven symbols and one winning line:

a) How many possible combinations of symbols can occur on the winning line? b) How many of these combinations contain three identical symbols? c) How many of these combinations contain exactly two identical symbols?

Solution:

a) Because there are seven symbols on each of the three reels, we have $7 \cdot 7 \cdot 7 = 343$ possible combinations.

b) For a specific symbol *S*, we have a single combination containing 3 *S*-symbols, namely (*SSS*). In total, we have $7 \cdot 1 = 7$ combinations.

c) For a specific symbol *S*, the combinations to be counted have the form:

(*SSx*), $x \neq S$, in number $1 \cdot 1 \cdot 6 = 6$

(*SxS*), $x \neq S$, in number $1 \cdot 1 \cdot 6 = 6$

(*xSS*), $x \neq S$, in number $1 \cdot 1 \cdot 6 = 6$.

Obviously, the order of elements has been taken into account.

We have no common combinations among these combinations (such a combination would have an (*SSS*) form, but $x \neq S$), so we are allowed to add the three numbers and we find 18. We have seven symbols; therefore, the total number of searched combinations is $7 \cdot 18 = 126$.

22) How many possible ways can three dice fall after they have been rolled (with respect to the numbers shown on their superior side)?

Solution:

The problem is similar to point a) of the previous problem (we use a die in place of a reel and the die numbers in place of symbols on the reel). Each die has six numbers, so the three dice can show $6 \cdot 6 \cdot 6 = 216$ possible combinations after rolling (one certain number on each die—the order does count).

……………….. missing part …………………………

Unsolved applications

1) Calculate: *C*(8, 3), *C*(11, 8), *C*(17, 5), *C*(19, 12), *C*(25, 7), *C*(41, 16), *C*(52, 43), *A*(7, 2), *A*(11, 7), *A*(15, 12), *A*(23, 4), *A*(31, 15), *A*(40, 11), *A*(47, 3).

2) In a 6/49 lottery, find:

a) the number of drawn variants possible containing numbers 5 and 17;

b) the number of drawn variants possible containing numbers 2, 7 and 10;

c) the number of drawn variants possible containing exactly three even numbers;

d) the number of drawn variants possible containing a minimum of three even numbers;

e) the number of drawn variants possible containing only number that are larger than 12;

f) the number of drawn variants possible containing at least three numbers larger than 15.

3) In a 5/40 lottery, find:

a) the number of possible played variants containing a minimum of three even numbers;

b) the number of drawn variants possible containing a minimum of three even numbers;

c) the number of possible variants played containing exactly two numbers less than 15;

d) the number of drawn variants possible containing exactly two numbers less than 15;

e) the number of possible played variants containing a minimum of two numbers less than 17;

f) the number of drawn variants possible containing a minimum of two numbers less than 19.

3) In a 52-card classical poker game, find:

a) the number of possible hands a player can be dealt that contain at least one *K* (king);

b) the number of possible hands a player can be dealt that contain exactly three ♦ symbols (diamonds);

c) the number of possible hands a player can be dealt that contain exactly three identical symbols (any);

d) the number of possible hands a player can be dealt that contain two pairs (any; no full house, no quads, but exactly two different pairs);

e) the number of possible hands a player can be dealt, only containing cards with a value larger than 9.

5) Same problem for a 32-card classical poker game.

6) In Texas Hold'em Poker, find the number of possible hands a player can be dealt, containing:

a) one 2 and one 5;

b) at least one 2 or at least one 5;

c) one less than 10 card and one larger than 10;

d) cards having different symbols (unsuited).

7) In Texas Hold'em, from the perspective of a neutral observer, calculate how many ways the five community cards can be distributed such that:

a) to contain exactly three diamonds (♦);
b) to contain a minimum of three diamonds;
c) to contain exactly three identical symbols (any);
d) to contain a minimum of three identical symbols;
e) to contain exactly three cards with same value;
f) to contain a minimum of three cards of same value (3 or 4);
g) to be all consecutive (for example: *A* 2 3 4 5 or 3 4 5 6 7);
h) to contain exactly four consecutive cards.

8) In Texas Hold'em, you are participating as a player and you were dealt 2♦ and J♠. How many ways can the flop cards be distributed so that they contain:
a) exactly two clubs;
b) a minimum of two clubs;
c) 2's or *J*'s (however many of them);
d) 2's and *J*'s (however many);
e) a pair of 2's (no triple);
f) a pair of *J*'s
g) a pair of 2's or a pair of *J*'s (no triple).

9) At a slot machine with five reels, eight symbols and one winning line:
a) How many combinations can occur on the winning line?
b) How many of them contain exactly three identical symbols (any)?
c) How many of them contain a minimum of three identical symbols (any)?

10) How many ways can four dice fall after being rolled? How many ways can four dice fall after being rolled, such that the sum of numbers shown is 17?

11) How many ways can a sport betting ticket be filled, if on the ticket are seven matches and the result of the bet is the total number of scored goals: 0 – 3, 4 – 6, 7 – 10 and >10 (four variants)?

BEGINNER'S CALCULUS GUIDE

Introduction

The previous chapters presented the probability notion in all its interpretations in lay language as well as at the mathematical and philosophical level.

As can be seen in these sections, no interpretation variant can omit the mathematical model given by probability theory.

When we speak about probability calculus, we refer strictly to the mathematical calculus tools provided by this theory, and exclude other subjective interpretations.

Although this calculus often assumes hypothetical approximations or pure calculation, in the end or even from the start, the numerical results thus obtained are much more relevant than any other subjective estimation based on intuition or on nonmathematical interpretations of probability.

This chapter is a guide to the calculus of numerical probabilities, and is structured so that it can be used by persons with a minimal mathematical background.

Although theoretically the guide can be studied and used without running through the mathematical chapter as a preliminary— because the formulas used in the applications are presented again for review before being applied and the presentation of solutions is algorithmic— we consider that a minimal knowledge of the classical definition of probability, probability field, operations with events and relations between events to be necessary.

Also, the combinatorial calculus represents a main tool that is used consistently in the applications presented, and understanding and applying this calculus requires reading the chapter titled *Combinatorics* before proceeding with these applications.

Besides this basic knowledge of probability theory and combinatorics, the only requirement for the reader is to have a good command of the four arithmetic operations between real numbers and of basic algebraic calculus.

These limitations in knowledge are possible because this guide deals only with finite or at the most discrete cases as the basis for these applications.

Most of the applications presented here come from games of chance, where we deal only with finite probability fields.

Theoretically, any probability calculus problem, no matter how complex, can be unfolded in successive elementary applications that use basic formulas, but most of the time finishing the calculus can be very laborious or even impossible, not to mention the high risk of the occurrence of errors during a long succession of calculations.

The use of combinatorics and even of classical probability repartitions can often solve such problems simply and elegantly, whereas the step-by-step approach is much too laborious and is predisposed to calculation errors.

If we composed a list of the minimal knowledge required by the reader who wants to solve finite probability applications by studying this guide, it would look as follows:

Previous background (from school):

– Operations with real numbers: addition, subtraction, multiplication, division, powers, order of operations, operations with fractions, reductions;

– Algebraic calculus: expanding the brackets, multiplication of expressions within brackets, raising an expression to a power, formulas of shortened calculus, factoring out, reduction of algebraic fractions;

Combinatorics knowledge (from school or the chapter titled Combinatorics*)*:

– Definition of permutations, arrangements and combinations;

– General formulas of permutations, arrangements and combinations;

– Combinatorial calculus procedures: properties (formulas), partitioning;

– Models of solved applications;

– Solving as many as applications as possible.

Basic probability knowledge (from school or the chapter titled Probability Theory Basics*)*:

– Operations with sets: intersection, union, difference, complementary;

– Events, operations with events as sets;

– Elementary events, incompatible events, independent events, mutually exclusive events;

– Classical definition of probability;

– Probability field.

Of course, the list can be extended depending on each reader's option; thus, we recommend that the wider the image of theoretical notions and results are, the surer and much better guarded against theoretical or calculation errors their application will be.

Inversely, the list can be reduced (but not by much) because the solution of many applications (especially those from the beginning) contain additional explanations about the notions involved in framing a problem and in the mathematical models used.

Besides this list of required mathematical knowledge, an important component of the ensemble of necessary skills is the ability to observe.

The correct framing of a problem, establishing the probability field to operate within and the relations between various events are a matter of solver's ability to observe.

This initial stage of the solving algorithm is essential in solving an application and finding the final numerical result.

But the ability to observe is not a native skill and does not result from a previous mathematical education. It can be acquired at any time through unceasing exercise.

This is also the reason for including in this guide a collection of solved applications with detailed explanations and instructions.

The complexity and difficulty of the applications grows progressively and their solutions follow exactly the general algorithm for solving presented in the next section.

The didactic goal of this guide is to enable the reader to solve any finite probability application alone or by consulting the guide.

At the end of this introduction, to achieve the best didactic results, we recommend that the reader do the following things:

– Read the sections that explain the probability notion in the chapters titled *What Is Probability?* (the sections on *Probability—the word, Probability as a limit, The probability concept*).

– Read at least once all definitions in the list of requirements of mathematical knowledge outlined earlier.

– Follow exactly the general algorithm for solving for any application and do not skip any stage, even if a particular application appears easy (do not pass directly to numerical calculus without framing the problem and establishing the probability field).

– Do revise the long or complex combinatorial calculations; this is where errors often occur.

– Do not try to apply a previous solving scheme to a similar application at any cost; the probability fields may be different, as can the events to be measured or the questions to be answered, even if their descriptions are similar or even identical.

– Try to memorize over time the formulas used to solve the applications.

– Revise the calculations and the entire solution algorithm each time the final numerical result seems too low or too high, but do not turn this into a general criterion for establishing the existence of an error (intuition can play bad tricks in probability).

– Do not approach the unsolved applications (except those recommended at the end of various sections) until you have run through the entire calculus guide and its solved applications.

– Do not stop your study if from the beginning you feel that solving the applications is beyond your comprehension; just take a break and resume reading later, when your concentration is higher. Read a paragraph several times if needed and come back to the information from previous chapters any time you consider it necessary.

The general algorithm of solving

Every solution of a probability application submits to a basic algorithm, which basically ensures the correctness of framing and approach to the calculus problem and of the application of the theoretical results as well.

Even though the methods of solving a problem can be multiple, all procedures are applied on the basis of this general algorithm, which is valid for any finite or discrete probability application.

The solution algorithm consists of three main stages:

1) *Framing the problem*

– Establishing the probability field attached to an experiment;
– Textually defining the events to be measured;
– Establishing the elementary events that are equally possible;
– Observing the independent, nonindependent and incompatible events;
– Necessary idealizations.

2) *Establishing the theoretical procedure*

– Choosing the solving method (step by step or condensed);
– Selecting the formulas to use.

3) *The calculus*

– Numerical calculus;
– Combinatorial calculus;
– Eventual approximations;
– Probability calculus (applying the formulas).

Framing the problem

This first stage of the solution algorithm is very important.

Although it does not include the probability calculus that is required by each application, it establishes the framework for this calculus by showing the optimal mathematical model that makes the correct application of theoretical results possible and ensures that relevant numerical results are acquired in the end.

We saw that the probability of an event makes sense from a mathematical point of view only if that event belongs to a Boolean structure, namely, a field of events.

If we consider the set Ω of all possible outcomes or the sample space of an experiment, then the set of events associated with that experiment is included in or equal to $\mathcal{P}(\Omega)$ and is a Boole algebra.

If we specified the set Ω and this set is finite or discrete, we have also specified the associated field of events.

Examples:

1) In the experiment involving tossing a coin, the sample space is $\Omega = \{H, T\}$ (H – heads, T – tails), and the field of events is $\mathcal{P}(\Omega) = \{\phi, \{H\}, \{T\}, \Omega\}$.

2) In the experiment involving tossing two coins, the sample space may be $\Omega = \{(H, T), (T, H), (H, H), (T, T)\}$ if we take into account the outcomes for each coin (we deal in this case with a set of ordered pairs), or may be $\Omega' = \{(HT), (HH), (TT)\}$, if we take into account the cumulative outcomes for both coins (it is a set of unordered pairs, namely combinations, in which order does not count).

Although the set Ω' stands for a set of outcomes that covers all possibilities, choosing the field of events $\mathcal{P}(\Omega')$ as basis for framing the application is not correct. As we will see further, the elementary events of this field cannot be considered equally possible.

In applications, specifying the sample space by enumerating its elements is not absolutely necessary because we are most interested in the number of these elements rather than in being able to see them.

3) Three persons are randomly chosen from a group of 100 with certain specified characteristics. The probability for at least one chosen person to have certain characteristics is required.

In such a problem, denoting all possible outcomes of the experiment and unfolding their set is useless. This set has only to be established and imagined.

The number of its elements is given by all 3-size combinations from 100 elements, namely, $C_{100}^{3} = 161700$.

Often, the field of events is not that easy to view, as can be seen in the next example. In these cases, the field of events must be rebuilt so that the event whose probability we are looking can belong to it.

……………….. missing part …………………………

Exercises and problems

1) At a European roulette (with thirty-seven numbers from 0 to 36), find the probability of occurrence of an uneven number after a spin.

Answer:

The sample space is {0, 1, 2, ..., 36}.

The events e_i ={occurrence of number i}, i = 0, ..., 36, are the elementary events of the experiment of spinning the roulette wheel.

They are equally possible (this is a necessary idealization). Each event has the probability 1/37.

The event to be measured is A – *occurrence of an uneven number*.

This is a compound event, which can be decomposed as $A = e_1 \cup e_3 \cup ... \cup e_{35}$ (in eighteen elementary events).

The elementary events are mutually exclusive; therefore, we have $P(A) = P(e_1) + P(e_3) + ... + P(e_{35}) = 18 \cdot 1/37 = 18/37 = 0.48648.$

In other words, from the total of thirty-seven equally possible outcomes, eighteen are favorable for the event A to occur, implying an 18/37 probability according to the classical definition of probability.

2) Two dice are rolled simultaneously. Calculate the probability for the sum of the points shown on both dice to be greater than 7.

Answer:

An elementary event is represented by a 2-size combination of numbers of the two dice. The set of elementary events is then $\{(a, b) \mid a \in \{1, 2, 3, 4, 5, 6\}, b \in \{1, 2, 3, 4, 5, 6\}\}$, which is a set of combinations with 6 x 6 = 36 elements.

a stands for the number shown on the first die and b for the number shown on the second.

Any such combination (a, b) is possible in the same measure.

The event to be measured is A: $a + b > 7$.

All the variants that are favorable for this inequality are:

2 + 6, 3 + 5, 3 + 6, 4 + 4, 4 + 5, 4 + 6, 5 + 3, 5 + 4, 5 + 6, 6 + 2, 6 + 3, 6 + 4, 6 + 5, 6 + 6, for a total of 14.

Observe that the order has been taken into account (both combinations $a + b$ and $b + a$ have been counted as different).

Each favorable combination is an elementary event and their union is event A.

We then have $P(A)$ = 14 x 1/36 = 14/36 = 7/18 = 0.38888.

An incorrect framing of this problem is one in which we would consider the set of elementary events as the set of possible sums of the points shown on dice: {sum 2, sum 3, sum 4, sum 5, sum 6, sum 7, sum 8, sum 9, sum 10, sum 11, sum 12}, with eleven elements.

Event A is the union of five elementary events (sum 8, sum 9, sum 10, sum 11 and sum 12).

Although it seems to be an easy choice and does not take into account the order (but the cumulative results of the two dice), it is not correct because the respective events cannot be considered equally possible.

For example, the event *sum 2* can occur in only one way (1 + 1, namely number 1 on both dice), while the event *sum 5* can occur in four ways (2 + 3, 3 + 2, 1 + 4, 4 + 1).

This makes impossible an *equally possible* type of idealization for these events.

In such a field of events, calculating the probability of event *A* as 5/11 is incorrect because the classical definition of probability is valid only for equally possible events. Keep this error example in mind.

……………….. missing part …………………………

Establishing the theoretical procedure

To establish the theoretical procedure through which an application is solved means choosing a solving method and selecting the mathematical formulas to be used in the probability calculus itself.

This selection is always made after the problem is properly framed. No formula can be applied without having first defined the ensemble of conditions (hypotheses) that match the mathematical model that generated the respective formula.

This is why respecting the chronological order of the two stages (framing the problem and establishing the theoretical procedure) is not just a recommendation, but a logical necessity.

Methods of solving

A probability calculus problem may have several solving methods (ways) that lead to the same correct result. This happens because probability theory is a consistent and rigorous theory from a mathematical point of view.

The numerous possible solving methods that use the basic theoretical results can be grouped into two main categories: the step-by-step methods and the condensed methods.

The solutions of finite type applications all use these solving methods, either individually or combined.

Obviously, depending on the application, other specific methods might appear that use more complex theoretical results, and these cannot be framed in the two main categories.

A solution might also use both methods in the various partial solutions that may be involved in a respective application.

The step-by-step method consists of the successive decomposition of the experimental ensemble into simpler individual tests, applying theoretical results to each part of the test and combining the partial results to obtain the probabilities of the events to be measured.

The condensed method consists of treating the experimental ensemble as a whole unit (one single test), in which the events to be measured are decomposed according to the field of events attached to the respective experiment and the theoretical results are applied directly to the events to be measured.

Condensed methods are specific to the usage of combinatorics or classical probability schemes.

To see how the two methods work, let us consider a few very simple applications.

Exercises and problems

1) Two dice are rolled. Find the probability for the first die to show 3 and the second to show 5.

A. *Step-by-step method*

We consider the rolling of the two dice as two separate experiments (tests): rolling the first die and rolling the second.

We assume these tests are independent.

Denote by *A – the first die shows 3* and by *B – the second die shows 5.*

The two experiments are the same type (rolling the die) and have the same sample space, so we can assume that the two attached fields of events are identical.

Thus, event $A \cap B$ *– the first die shows 3 and the second die shows 5* make sense with respect to the intersection operation.

We have $P(A) = 1/6$, $P(B) = 1/6$ (we have applied the classical definition of probability for each part of the test) and, because A and B are independent events, we have

$P\left(A \cap B\right) = P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{6} = 1/36$ (this is the selected and applied formula).

B. *Condensed method*

We consider the rolling of the two dice as one single experiment.

The sample space attached to this experiment is the set of ordered pairs (a, b), with $a, b \in \{1, 2, 3, 4, 5, 6\}$.

Their number is $6 \times 6 = 36$ and their occurrences are equally possible elementary events. Among them, a single pair is favorable for event E – *the first die shows 3 and the second die shows 5* to occur, namely the pair (3, 5).

We then have, according to the classical definition of probability, that $P(E) = 1/36$.

Observe that in the solution using the condensed method there was no longer a need for the additional formula of the probability of the intersection of two independent events, the classical definition of probability being sufficient.

2) A player participating in a card game with a 52-card deck is dealt two cards, with no cards in view at that moment. What is the probability of the player being dealt A♣ and 7♥? What is the probability of the player being dealt an *A* and a 7?

A. *Step-by-step method*

We consider the distribution of the two cards as two different experiments (the distribution of the first card and then the distribution of the second card, both performed by a dealer).

Denote the following events:

A – player is dealt an A♣ as the first card

B – player is dealt a 7♥ after the A♣ is dealt

C – player is dealt a 7♥ as first card

D – player is dealt an A♣ after the 7♥ is dealt

Events *A* and *C* belong to the field of events that is attached to first experiment, while *B* and *D* belong to the field of events attached to the second experiment.

Although this problem deals with two different probability fields, by identifying the events with the set of outcomes through which they occur, we see that these events are all part of the sample space of the first experiment; therefore, the intersection and union operations between them make sense.

We know that events A and B are independent, as are C and D, while A and C are incompatible.

Thus, events $A \cap B$ and $C \cap D$ are also incompatible.

$P(A) = 1/52$, $P(B) = 1/51$, $P(C) = 1/52$, $P(D) = 1/51$ (from the classical definition of probability).

The event to be measured is *player is dealt A♣ and 7♥* – $(A \cap B) \cup (C \cap D)$ and we have:

$$P\big((A \cap B) \cup (C \cap D)\big) = P(A \cap B) + P(C \cap D) =$$

$$P(A)P(B) + P(C)P(D) = \frac{1}{52} \cdot \frac{1}{51} + \frac{1}{52} \cdot \frac{1}{51} = 1/1326 .$$

We have applied the formula of probability of intersection of two independent events and the formula of probability of union of two incompatible events.

B. *Condensed method*

We consider one experiment, namely, the distribution of two cards performed by dealer.

The set of elementary events attached to this experiment can be identified with the set of 2-size combinations from the 52, which has $C_{52}^2 = 1326$ elements.

Among them, a single combination is favorable for the occurrence of event to be measured *player is dealt A♣ and 7♥,* so its probability is $\dfrac{1}{C_{52}^2} = 1/1326$.

For the second question (the probability of a player being dealt any A and any 7):

A. *Step-by-step method*

We consider the distribution of the two cards as two different experiments (the distribution of the first card and then the distribution of the second card, both performed by a dealer).

Denote the following events:

………………. missing part ……………………….

2) You buy a ticket (one variant) for the 6/49 lottery. What is the probability of exactly three numbers printed on your ticket being drawn?

Answer:

Solution 1

Let (*abcdef*) be the played variant (*a*, *b*, *c*, *d*, *e*, *f* are distinct natural numbers from 1 to 49).

Let us denote the events:

A_{abc} – exactly three numbers from your ticket are drawn: the first, the second and the third, namely, *a*, *b*, *c*;

A_{abd} – exactly three numbers from your ticket are drawn: the first, the second and the fourth, namely, *a*, *b*, *d*; and so on;

A_{def} – exactly three numbers from your ticket are drawn: the fourth, the fifth and the sixth, namely, *d*, *e*, *f*;

We have $C_6^3 = \dfrac{4 \cdot 5 \cdot 6}{2 \cdot 3} = 20$ such events A_{ijk} and the event to be measured is their union: $\bigcup\limits_{\substack{i,j,k \in \{a,b,c,d,e,f\} \\ i<j<k}} A_{ijk}$.

The total number of possible combinations that can be drawn is C_{49}^6 .

The probability of an event A_{ijk} can be calculated very simply: the combinations that are favorable to event A_{ijk} are of the form (*ijkxyz*), with *x*, *y*, *z* distinct and different from *a*, *b*, *c*, *d*, *e*.

(*ijkxyz*) = (*ijk*)(*xyz*)

Their number is $1 \cdot C_{49-6}^3 = C_{43}^3$ (one combination (*ijk*) and *x*, *y*, *z* may take 49 – 6 = 43 values).

According to (F3), we then have: $P\left(A_{ijk}\right) = \dfrac{C_{43}^3}{C_{49}^6}$, for any *i*, *j*, *k*.

Observe that events A_{ijk} are mutually exclusive; therefore, we are allowed to apply (F6): $P\left(\bigcup_{\substack{i,j,k\in\{a,b,c,d,e,f\}\\ i<j<k}} A_{ijk}\right)=20\cdot P\left(A_{ijk}\right)=\frac{20C_{43}^{3}}{C_{49}^{6}}$.

Exercise: Do the complete calculation and express the probability as a percentage.

Solution 2

This experiment corresponds to the scheme of the nonreturned ball, with the following equivalencies:

– draw of a ball = draw of a number; the experiment is repeated for n = 6 times, without putting back the ball;

– we have two colors (s = 2): the numbers (balls) from the ticket (a, b, c, d, e, f) represent the first color, c_1 , and the remaining 49 – 6 = 43 balls have the second color, c_2;

– we have $a_1=6$ balls of color c_1 and $a_2=43$ balls of color c_2;

The problem asks for the probability of $\alpha_1=3$ balls of color c_1 and $\alpha_2=3$ balls of color c_2 being drawn.

By applying (F30), we find the searched probability to be $P\left(6;3,3\right)=\frac{C_6^3C_{43}^3}{C_{49}^6}$.

………………. missing part ……………………….

PROBABILITY CALCULUS APPLICATIONS

Because skills in probability calculus and in correctly applying the theoretical results are acquired only through exercise, this chapter contains a collection of solved and unsolved applications that cover most of the range of classical probability problems.

The reader who has studied the *Beginner's Calculus Guide* can practice the theoretically acquired skills by studying the solutions to the classical problems and solving as many applications as possible.

The proposed problems involve probability calculus applications for finite cases. Their difficulty grows progressively from simple to the intermediate level.

We also recommend that readers who want to improve their application and probability calculus skills not limit themselves to the sections of applications in this book, but work on other specific problem books, too.

Solved applications

1) Determine $\mathcal{P}(\Omega)$ if:

a) $\Omega=\{1,2\}$;

b) $\Omega=\{A\}$;

c) $\Omega=\{\{0,1\},\{2,3\},\{4\}\}$.

<u>Solution:</u>

a) $\mathcal{P}(\Omega)=\{\phi,\{1\},\{2\},\{1,2\}\}$;

b) $\mathcal{P}(\Omega)=\{\phi,\{A\}\}$;

c) $\mathcal{P}(\Omega)=$

$\{\phi,\{\{0,1\}\},\{\{2,3\}\},\{\{4\}\},\{\{0,1\},\{2,3\}\},\{\{0,1\},\{4\}\},\{\{2,3\},\{4\}\},\Omega\}$

2) Write the sample space for the following experiments:

a) draw of a ball from an urn containing seven balls;

b) draw of two balls from two urns (one ball from each), the first containing three green balls and the second two red balls;

c) draw of a card from a 24-card deck (from the 9 card upward);

d) rolling two dice;

e) choosing three numbers from the numbers 1, 2, 3, 4, 5;

f) choosing seven letters from the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*;

<u>Solution:</u>

a) By numbering the balls, we have:

$\Omega=$ {ball 1, ball 2, ball 3, ball 4, ball 5, ball 6, ball 7} or, equivalent, $\Omega=\{1,2,3,4,5,6,7\}$.

b) By numbering the balls from the two urns and denoting by *g* a green ball and by *r* a red ball, the sample space is the following set of ordered pairs:

$\Omega=\{(1g,1r),(1g,2r),(2g,1r),(2g,2r),(3g,1r),(3g,2r)\}$.

c) $\Omega=$ {9♠, 9♣, 9♥, 9♦, 10♠, 10♣, 10♥, 10♦, J♠, J♣, J♥, J♦, Q♠, Q♣, Q♥, Q♦, K♠, K♣, K♥, K, A♠, A♣, A♥, A♦}.

d) The sample space is the following set of ordered pairs:
$\Omega = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$.

e) The sample space is the set of all 3-size combinations of numbers from the 5 given:
$\Omega = \{(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4), (1, 3, 5), (1, 4, 5), (2, 3, 4), (2, 3, 5), (2, 4, 5), (3, 4, 5)\}$.

f) The sample space is the set of all 7-size combinations of letters from the 8 given:
$\Omega = \{(a, b, c, d, e, f, g), (a, b, c, d, e, f, h), (a, b, c, d, e, g, h), (a, b, c, d, f, g, h), (a, b, c, e, f, g, h), (a, b, d, e, f, g, h), (a, c, d, e, f, g, h), (b, c, d, e, f, g, h)\}$.

3) Find the number of all possible outcomes for the following experiments:

a) rolling three dice; generalization: rolling n dice;

b) spinning a slot machine with four reels having eight symbols each; generalization: spinning a slot machine with n reels of m symbols each;

c) dealing a player three cards from a 52-card deck;

d) dealing two players two cards each from 50 cards;

e) a race with nine competitors.

Solution:

a) Rolling a die has six possible outcomes, so rolling the three dice has $6 \times 6 \times 6 = 216$ possible outcomes.

b) Spinning a reel has eight possible outcomes, so spinning the four reels has $8 \times 8 \times 8 \times 8 = 4096$ possible outcomes.

Generalization: we have $\underbrace{m \times m \times ... \times m}_{n} = m^n$ possible outcomes.

c) The result is given by the number of 3-size combinations from 52, namely, $C_{52}^{3} = \dfrac{50 \cdot 51 \cdot 52}{2 \cdot 3} = 22100$.

d) The two players are dealt a double card combination (*xy*)(*zt*) from the 50 cards. The result is given by the number of these combinations, which is $C_{50}^2 C_{48}^2 = \frac{49 \cdot 50}{2} \cdot \frac{47 \cdot 48}{2} = 1381800$.

e) The result is given by the number of permutations of nine elements, namely, $9! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 = 362880$.

4) Write the field of events attached to the following experiments:

a) tossing a coin;

b) drawing a ball from an urn with three balls;

c) drawing two balls from an urn with three balls.

Solution:

a) The sample space is $\Omega = \{H, T\}$ (H – heads, T – tails), so the field of events is $\Sigma = \mathcal{P}(\Omega) = \{\phi, \{H\}, \{T\}, \{H, T\}\}$.

b) Denoting the three balls by *a*, *b*, *c*, we have $\Omega = \{a, b, c\}$ and

$\Sigma = \mathcal{P}(\Omega) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

c) With the same denotations as from point b), we have:

$\Omega' = \{(a, b), (a, c), (b, c\}$ and $\Sigma' = \mathcal{P}(\Omega') =$

$\{\phi, \{(a,b)\}, \{(a,c\}, \{(b,c)\}, \{(a,b),(a,c)\}, \{(a,b),(b,c)\}, \{(a,c),(b,c)\}, \Omega'\}$

5) An urn contains thirty balls numbered 1 to 30. Relating to the experiment of drawing a ball, what can you state about the following events (elementary, compound, relations between them):

A – the number of the drawn ball is even;

B – the number of the drawn ball is a multiple of 4;

C – the number of the drawn ball is 5;

D – the number of the drawn ball is a multiple of 5;

E – the number of the drawn ball is a power of 5.

Solution:

The only elementary event is *C*; the other events are compound.

The following couples of events are incompatible: *A* and *C*, *B* and *C*. We have the following inclusions: $B \subset A$, $E \subset D$, $C \subset D$, $C \subset E$.

6) Two cards are drawn from a 52-card deck. Consider the events:

A – two aces are drawn;

B – two cards higher than *Q* are drawn;

C – 5♦ and a *J* are drawn.

Decompose these events in elementary events and specify their numbers.

Solution:

The elementary events attached to this experiment are the occurrences of the 2-size combinations (couples) from the 52 cards.

A = {(*AA*)} = {(A♠A♣), (A♠A♣), (A♠A♦), (A♣A♥), (A♣A♦), (A♥A♥)}.

The set *A* has $C_4^2 = 6$ elements (elementary events).

B = {(*xy*), with *x*, *y* being *K* or *A*} = {(A♠A♣), (A♠A♣), (A♠A♦), (A♣A♥), (A♣A♦), (A♥A♥), (K♠K♣), (K♠K♣), (K♠K♦), (K♣K♥), (K♣K♦), (K♥K♥), (K♠A♠), (K♠A♣), (K♠A♥), (K♠A♦), (K♣A♠), (K♣A♣), (K♣A♥), (K♣A♦), (K♥A♠), (K♥A♣), (K♥A♥), (K♥A♦), (K♦A♠), (K♦A♣), (K♦A♥), (K♦A♦)}.

The set *B* is the union of the mutually exclusive sets {(*KK*)}, {(*AA*)} and {(*AK*)} and has $C_4^2 + C_4^2 + 4 \cdot 4 = 28$ elements.

C = {(5♦ *J*)} = {(5♦ J♠), (5♦ J♣), (5♦ J♥), (5♦ J♦)}.

C has four elements.

……………….. missing part …………………………

24) At a blackjack game, calculate the probability for a player to get a total of twenty points from the first two cards (provided no other cards are shown), in two cases: a) a 52-card deck is used; b) two 52-card decks are used.

Solution:

a) for one deck

The variants totaling twenty points are of the type *A* + 9 or 10 + 10 (as a value; that is, any 2-size combination of cards from 10, *J*, *Q*, *K*). We have sixteen variants *A* + 9 (4 aces and 4 nines) and $C_{16}^2 = 120$ variants 10 + 10 (all 2-size combinations of cards from the sixteen cards with a value of 10). The number of all possible distribution variants for two cards is $C_{52}^2 = 1326$. The probability is then $P = \dfrac{16+120}{1326} = 68/663$.

b) for two decks

The variants of type A + 9 number sixty-four and those of type 10 + 10 (as value) number $C_{32}^2 = 496$. The number of all possible distribution variants is $C_{104}^2 = 5356$. The probability is then

$P = \dfrac{64+496}{5356} = 140/1339.$

25) You are participating in a Texas Hold'em Poker game with seven opponents and you are dealt (K3). Calculate the probability of your opponents holding no K (before the flop).

<u>Solution:</u>

We calculate the probability of the contrary event

A – at least one opponent holds at least one K.

We first calculate the probability of a fixed opponent holding at least one K. The combinations that are favorable for this event are (Kx), with x different from the cards in your hand. To count them, we split them into two groups:

(Kx), x different from K – and number 3 x (52 – 2 – 1 – 2) = 3 x 47 = 141 , and

(KK) – and number $C_3^2 = 3$.

In total, we have 141 + 3 = 144 favorable combinations, from $C_{50}^2 = 1225$ possible, so the probability is 144/1225.

Denoting by A_i the events *opponent number i holds at least one K* (i = 1, ..., 7), we have $P(A_i) = 144/1225$ and can then study the intersections of 2, 3, 4,5, 6 and 7 sets A_i:

There are three K-cards still in play, so more than three opponents cannot hold at least one K. Thus, the intersections of more than three sets A_i are empty. We have $C_7^2 = 21$ intersections for two sets A_i. Such intersection contains double combinations of the type (Kx)(Ky), with x and y different from the cards in your hand. We split them into three groups:

$(Kx)(Ky)$, with x and y different from K – and number
$3 \times (52 - 2 - 1 - 2) \times 2 \times (52 - 2 - 2 - 1 - 1) = 12972$;

$(KK)(Ky)$, with y different from K – and number
$C_3^2 \times 1 \times (52 - 2 - 2) = 144$, and

$(Kx)(KK)$, with x different from K – and number
$3 \times (52 - 2 - 1 - 1) \times C_2^2 = 144$.

In total, we have 12972 + 144 + 144 = 13260 combinations that are favorable from $C_{50}^2 C_{48}^2$ possible, so $P\left(A_i \cap A_j\right) = \dfrac{13260}{C_{50}^2 C_{48}^2}$,

for any $i, j = 1, ..., 21, i \neq j$.

We have $C_7^3 = 35$ intersections of three sets A_i.

Such intersection contains triple combinations of the type $(Kx)(Ky)(Kz)$, with x, y and z different from K, and number
$3 \times (52 - 2 - 1 - 2) \times 2 \times (52 - 2 - 2 - 1 - 1) \times 1 \times$
$\times (52 - 2 - 2 - 2 - 1) = 583740$.

The number of all possible triple combinations is $C_{50}^2 C_{48}^2 C_{46}^2$, so we have $P\left(A_i \cap A_j \cap A_k\right) = \dfrac{583740}{C_{50}^2 C_{48}^2 C_{46}^2}$, for any
$i < j < k,\ i, j, k = 1, ..., 35$.

We can now apply the inclusion-exclusion principle:

$$P(A) = 7 \cdot \frac{144}{1225} - 21 \cdot \frac{13260}{C_{50}^2 C_{48}^2} + 35 \cdot \frac{583740}{C_{50}^2 C_{48}^2 C_{46}^2} = 0.63562.$$

The probability of the contrary event (requested by the problem) is $1 - P(A) = 0.36438$.

………………... missing part …………………………

29) You are participating in a Texas Hold'em Poker game with n opponents, you are dealt two suited cards (cards with the same symbol) and the flop comes with three additional cards of your suit.

Find the general formula of probability of none of your opponents holding two cards of your suit.

Solution:

We have 52 – 2 – 3 = 47 unseen cards.

There are still 13 – 2 – 3 = 8 cards of your suit among the unseen cards.

Let us denote by S the symbol of your suit.

We find first the probability of a specific opponent holding (SS) (denote this event by A):

The total number of possible 2-size combinations that an opponent can be dealt is $C_{47}^2 = 1081$.

The number of favorable combinations (SS) is $C_8^2 = 28$.

The probability is then $P(A) = C_8^2 / C_{47}^2 = 28/1081$.

Denote by A_i the events *opponent number i holds (SS),* $i = 1, ..., n$. The event to be measured is $\bigcup_{i=1}^{n} A_i$.

We have $C_n^2 = n(n-1)/2$ intersections of two events A_i.

Such an intersection contains the double combinations (SS)(SS), and number $C_8^2 C_6^2$. The number of all double combinations two opponents can be dealt is $C_{47}^2 C_{45}^2$.

Thus, $P\left(A_i \cap A_j\right) = \dfrac{C_8^2 C_6^2}{C_{47}^2 C_{45}^2}$, for any $i \neq j$, $i, j = 1, ..., n$.

We have $C_n^3 = n(n-1)(n-2)/6$ intersections for each of three events A_i. Such an intersection contains the triple combinations (SS)(SS)(SS), and numbers $C_8^2 C_6^2 C_4^2$.

The number of all triple combinations three opponents can be dealt is $C_{47}^2 C_{45}^2 C_{43}^2$.

Thus, $P\left(A_i \cap A_j \cap A_k\right) = \dfrac{C_8^2 C_6^2 C_4^2}{C_{47}^2 C_{45}^2 C_{43}^2}$, for any $i < j < k$, $i, j, k = 1, ..., n$.

We have $C_n^4 = n(n-1)(n-2)(n-3)/24$ intersections of four events A_i. Such an intersection contains the quadruple combinations (SS)(SS)(SS)(SS), and numbers $C_8^2 C_6^2 C_4^2 C_2^2 = C_8^2 C_6^2 C_4^2$.

The number of all quadruple combinations four opponents can be dealt is $C_{47}^2 C_{45}^2 C_{43}^2 C_{41}^2$.

Thus, $P\left(A_i \cap A_j \cap A_k \cap A_h\right) = \dfrac{C_8^2 C_6^2 C_4^2}{C_{47}^2 C_{45}^2 C_{43}^2 C_{41}^2}$, for any $i < j < k < h$, $i, j, k, h = 1, ..., n$.

A maximum of four opponents can simultaneously hold (*SS*), because there are only 8 *S*-cards in play; therefore, the intersections of more than four events A_i are empty.

We can now apply the inclusion-exclusion principle:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \frac{nC_8^2}{C_{47}^2} - \frac{C_n^2 C_8^2 C_6^2}{C_{47}^2 C_{45}^2} + \frac{C_n^3 C_8^2 C_6^2 C_4^2}{C_{47}^2 C_{45}^2 C_{43}^2} - \frac{C_n^4 C_8^2 C_6^2 C_4^2}{C_{47}^2 C_{45}^2 C_{43}^2 C_{41}^2}$$

and this is the searched formula.

Exercise: Do the complete combinatorial calculus and do the algebraic calculus to put the above expression in a polynomial form.

Unsolved applications

1) Determine $\mathcal{P}(\Omega)$ if:

a) $\Omega = \{0, 1, 2\}$;

b) $\Omega = \{A, B\}$;

c) $\Omega = \{\{a, b\}, \{c, d\}, \{a, b, c\}, \{e\}\}$.

2) Write the sample space of the following experiments:

a) draw of a ball from an urn containing eight balls;

b) draw of two balls from two urns (one ball from each), the first containing four yellow balls and the second three black balls;

c) draw of a card from a 32-card deck (from 7 upward);

d) rolling three dice;

e) choosing four numbers from among the numbers 5, 7, 9, 11, 13, 15;

f) choosing eight letters from among the letters *m*, *n*, *o*, *p*, *q*, *r*, *s*, *t*, *u*, *v*;

3) Find the number of all possible outcomes for the following experiments:

a) tossing five coins; generalization: tossing *n* coins;

b) spinning a slot machine with five reels and ten symbols;

c) dealing a player four cards from a 52-card deck;

d) dealing two players two cards each from 52 cards;

e) dealing three players three cards each from 52 cards;

f) a race with ten competitors.

4) Write the field of events attached to the following experiments:

a) choosing a card from a 24-card deck;

b) drawing a ball from an urn with five balls;

c) drawing five balls from an urn with seven balls.

5) An urn contains fifty balls numbered 1 to 50.

Relating to the experiment of drawing a ball, what can you state about the following events (elementary, compound, relations between them):

A – the number of the drawn ball is even;
B – the number of the drawn ball is a multiple of 4;
C – the number of the drawn ball is 5;
D – the number of the drawn ball is a multiple of 5;
E – the number of the drawn ball is a power of 5;
F – the number of the drawn ball is a multiple of 10;
G – the number of the drawn ball is a multiple of 3;
H – the number of the drawn ball is a power of 3;
I – the number of the drawn ball is even.

6) Two cards are drawn from a 52-card deck. Consider the events:

A – two clubs are drawn;
B – two cards having a value less than 5 are drawn;
C – a 7 and a Q are drawn.

Decompose these events in elementary events and specify their numbers.

7) Find the probability of getting a multiple of 3 at a die roll.

Find the probability of getting a total of 5 points when rolling two dice. Find the probability of getting a total of 10 points when rolling three dice.

8) An urn contains nine white balls and four black balls. Find the probability of the following events:

a) A – drawing a white ball;
b) B – drawing a black ball.

9) In a pencil box are five pairs of pencils of same length (five separate lengths). Two pencils are randomly drawn from the box.

What is the probability of drawing a pair of pencils of the same length?

10) In a student's briefcase are seven math books, two English books and two drawing books. What is the probability that a randomly chosen book is a math book?

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69) A student must sit for an examination consisting of four questions selected randomly from a list of 150 questions.

To pass, the student must answer all four questions.

What is the probability that the student passes the examination if he or she knows the answers to 100 questions on the list?

Generalize this.

70) n shooters simultaneously fire at a mobile target.

The probability of hitting the target is the same for all shooters and is equal to $1/k$, where k is a non-negative natural number.

Calculate the probability that the target is hit by at least one shooter.

71) From an urn that contains n white and m black balls, k balls are drawn at random. What is the probability that there are r ($r \leq n$) white balls among them?

72) We have four urns; the first contains five white and four black balls, the second contains three white and six black balls, the third contains two white and five black balls and the fourth contains two white and three black balls.

A ball is drawn from a randomly chosen urn. Find the probability of the drawn ball being black.

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102) (De Mere's paradox) Prove that to obtain at least one 1 at a throw of four dice is more probable than to obtain, at least once, two 1's at twenty-four throws of two dice.

103) n points divide a circle into equal circular arcs.

Two points are randomly chosen from them. What is the mean of the distance between the chosen points?

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